CARNEGIE INSTITUTION OF WASHINGTON

PUBLICATION No. 408

1930

THERMODYNAMIC RELATIONS IN MULTI-COMPONENT SYSTEMS

BY

ROY W. GORANSON

Physicist, Geophysical Laboratory,
Carnegie Institution of Washington

PUBLISHED BY CARNEGIE INSTITUTION OF WASHINGTON

JULY, 1930

WASHINGTON TYPOGRAPHERS, INC.
WASHINGTON, D. C.

PREFACE

A large part of the work at the Geophysical Laboratory consists of experimental thermodynamics. This means that we are continually evaluating explicitly the thermodynamic relations which exist between the variables of particular systems, these variables including temperature, pressure, and amounts of the substances that determine their composition. It is therefore very desirable, and for this reason work was begun, to evaluate as many as possible of the relations existing between the variable quantities of multicomponent systems in terms of quantities that can be readily obtained from experiment, and tabulate them in a compact and easily accessible form.

In order to do this it was found necessary to make a skeleton outline of the whole structure of thermodynamics. Gibbs developed this subject but his treatment was couched in the mathematical language of his day which had not then been developed as a tool for the physicist and consequently readers found difficulty in deciphering it. Later writers went to the other extreme in avoiding mathematical language as much as possible. Now statements by such writers must necessarily be incomplete or ambiguous for physics has progressed to such an extent that physicists find ordinary language too poor to express the precise delicate shades of meaning that are found to be necessary. Mathematics was therefore developed to serve this purpose. Hence it was suggested that the outline that was constructed and used here should be completed very fully and made to serve as text for the tables.

I am indebted to George Tunell of this Laboratory who is in a large measure responsible for the existence of the book by giving lavishly of his time and effort in constructing the framework of this thesis, and in particular for his work in chapters one to three. I also wish to thank P. W. Bridgman of Harvard University and L. H. Adams of this Laboratory for reading and criticizing the manuscript.

ROY W. GORANSON

July, 1929.
Geophysical Laboratory
Washington, D. C.

CONTENTS

CHAPTER I. FUNDAMENTAL IDEAS

Undefined concepts or directly measurable quantities

CHAPTER II. SIMPLE HOMOGENEOUS SYSTEMS

Unit mass systems

ART. CHAPTER II—Continued PAGE

Homogeneous one-component systems of variable mass

CHAPTER III. HOMOGENEOUS BINARY SYSTEMS OF VARIABLE MASS AND COMPOSITION

CHAPTER IV. HOMOGENEOUS N-COMPONENT SYSTEMS

CHAPTER V. STRAIN

ART. CHAPTER V—Continued PAGE

CHAPTER VI. STRESS

CHAPTER VII. THERMODYNAMIC TREATMENT OF SYSTEMS HOMOGENEOUSLY STRAINED

CHAPTER VIII. THE STRESS-STRAIN RELATIONS FOR ISOTHERMAL CHANGES OF STATE

ART. CHAPTER VIII—Continued PAGE

CHAPTER IX. SYSTEMS NOT IN EQUILIBRIUM. IRREVERSIBLE PROCESSES

CHAPTER X. INTRODUCTION TO THE TABLES OF THERMODYNAMIC RELATIONS

Part 2

INTRODUCTORY

The science of thermodynamics deals with work and heat. Since all physical and chemical processes, which thereby include all natural phenomena, involve work and heat, it is apparent that thermodynamics is a fundamental and far-reaching science.

It was therefore thought desirable to have as many thermodynamic relations as possible in multi-component systems readily available, especially since a large part of the work done at the Geophysical Laboratory consists of experimental thermodynamics. This idea grew out of a study of the condensed collection of thermodynamic formulas derived by Professor Bridgman[1] for one-component systems of one and two phases and constant mass (the three phase one-component system not being variable). Bridgman also indicates the extension of his tables to more complex cases of one-component systems in which electrical forces, surface tension, and other forces are present in addition to hydrostatic pressure.

Bridgman's tables refer to systems of constant mass, but it is well known that all of the functions for the one-component system of variable mass may be computed from the unit mass functions. It is proved herein that the variable mass functions for multi-component systems may also be readily computed from the unit mass functions alone without the necessity of any additional experimental measurements.

For the purposes of experimental thermodynamics it is highly desirable if not essential that the quantities necessary to be directly measured in order to formulate a fundamental equation be known. Bridgman has already stated the quantities necessary to be measured for formulating the fundamental equation for the one-component single phase system. I have done this for the multi-component systems in my introduction to the tables.

How complete it is desired to make a mathematical treatment, *i. e.*, of what length to make the steps between the equations,

[1] P. W. Bridgman, A Condensed Collection of Thermodynamic Formulas (Harvard University Press), 1925.

depends both on the use to which it is to be put and upon the mathematical facility and intuition of the user. For example, in publishing his papers on Heterogeneous Equilibria, Gibbs omitted so many of the intervening steps between his equations that many of his readers have found difficulty in following his development of the subject. For this reason it was suggested that the derivations given here be treated very fully and completely.

Thermodynamics begins with certain undefined physical concepts (directly measurable quantities) and certain unproved hypothetical relations between them (physical hypotheses). All of the other concepts (variable quantities) of the science are defined in terms of the initial undefined concepts and all of the theorems of the science are deduced from the definitions and the initial physical hypotheses.[1]

The physical hypotheses of thermodynamics can of course be stated in words without the use of mathematical symbols. In order to deduce theorems from the physical hypotheses it is necessary to express the hypotheses symbolically by means of mathematical equations. Furthermore they can be stated with the same clarity and much more briefly by means of the mathematical equations of partial derivatives and line integrals. On this point Poincaré[2] says: "All laws are deduced from experiment; but to enumerate them, a special language is needful; ordinary language is too poor, it is besides too vague, to express relations so delicate, so rich, and so precise. This therefore is one reason why the physicist can not do without mathematics; it furnishes him the only language he can speak." In most of the problems independent variables of two orders are present and for dealing with independent variables of two orders[3] the methods of partial derivatives and line integrals were especially designed. As Professor W. F. Osgood has said, "In thermodynamics a thoroughgoing appreciation of what the independent variables are (in order that, when the letters expressing the variables of the two classes overlap the meaning of

[1] See Warren Weaver, American Math. Monthly 36, 1929, p. 39.
[2] H. Poincaré, The Foundations of Science, 1921, p. 281.
[3] This is the terminology used by W. F. Osgood, Advanced Calculus, 1925, pp. 115, 140. Another phraseology used by mathematicians is "functions of functions."

the partial derivatives may be clear) and the ability to think in terms of line integrals, are indispensable."

While this treatment is an attempt to begin with the undefined concepts, definitions and physical hypotheses and from them develop the subject in a mathematically rigorous manner, mathematical rigor has not been pursued for its own sake.

On the broad subject of the relationships between physics and mathematics Courant[1] has written:

"From time immemorial mathematics has derived powerful impulses from the close relationships which exist between the problems and methods of analysis and the perceptual concepts of physics. For the first time in the last decade a crumbling away of this connection has taken place in that mathematical investigation in many cases cut loose from its perceptual starting points and especially in analysis often concerned itself all too exclusively with refinement of its methods and sharpening of its concepts. Many students of analysis have thus lost a full knowledge of the close connection of their science with physics and other subjects while on the other hand the physicists have lost the understanding of the problems and methods of the mathematicians, in fact even of the mathematicians' entire sphere of interest and language. Without doubt in this tendency lies an important threat for science; the current of scientific development is in danger of seeping out further and further to ooze away and dry up. If it shall avoid this fate we must direct a good part of our forces toward again uniting the separated parts in developing clearly by means of collective points of view the internal connections of multifarious facts. Only thus will a real mastery of the materials be possible for the student and the ground be prepared for the investigator leading to a further organic development." In any case the question is merely whether rigorous mathematics constitutes a more useful tool than "non-rigorous" mathematics and this question probably can not be answered from *a priori* considerations. At the same time the term "non-rigorous" can only mean incomplete or incorrect. The degree of completeness desirable is

[1] Author's translation from the Preface of Methoden der Mathematischen Physik I, R. Courant und D. Hilbert, Julius Springer, Berlin, 1924.

of course a matter of judgment. As to the second interpretation of the term "non-rigorous," while the whole question must of course be left to the empirical test, it is hard to see how incorrect mathematics could be more useful than correct mathematics.

Of course no physical treatment can be more accurate than the initial hypotheses. Hence no treatment of science can be made rigorously accurate. For example, all physical hypotheses and definitions in this treatment presuppose that our physical operations on systems are carried out on a certain scale. It is believed that all matter can be subdivided into electrons and protons. Now suppose that we are able to observe and wish to treat the behavior of these electrons and protons in a system. All we can say is that this treatment does not apply to such dynamic systems since the variables used in this treatment would not uniquely define the internal conditions of such a system. We therefore definitely limit ourselves to systems composed of such a large aggregate of these electrons and protons that when any such system is considered as a whole, for we do not concern ourselves with the internal condition of the system, the state of this system is uniquely determined by the set of operations we limit ourselves to, which is, for any one system, the characteristic equation or equation of state we set down for it.

As a résumé, it may be stated that the following have been the principal aims of this treatment:

(1) Assuming the above limitations of the subject, to begin with the undefined concepts and physical assumptions and present the science of classical thermodynamics in a logical and mathematically rigorous manner.

(2) To fill in the gaps existing in the present literature by deducing the theorems necessary for this development.

(3) To evaluate the mathematical functions in terms of directly measurable quantities.

(4) To compute the mathematical relationships obtainable between the variables.

NOMENCLATURE

t	temperature in degrees on the Centigrade scale.
θ	temperature in degrees on the absolute thermodynamic scale.
p	pressure in dynes per square centimeter or baryes.
$\mathbf{m}_k$	mass in grams of component k.
m_k	mass fraction of component k, $= \dfrac{\mathbf{m}_k}{\mathbf{m}_1 + \ldots + \mathbf{m}_n}$
$\mathbf{v}$	total volume in cubic centimeters.
v	specific volume, volume per unit mass, in cubic centimeters per gram.
$\boldsymbol{\varepsilon}$	total internal energy in dyne centimeters
ϵ	internal energy per unit mass in dyne centimeters per gram.
$\mathbf{n}$	total entropy in dyne centimeters per degree.
η	entropy per unit mass in dyne centimeters per degree per gram.
$\boldsymbol{\zeta} = \boldsymbol{\varepsilon} + p\,\mathbf{v} - \theta\,\mathbf{n}$	total zeta (Lewis' free energy F) in dyne centimeters.
$\zeta = \epsilon + p\,v - \theta\,\eta$	zeta per unit mass in dyne centimeters per gram.
$\boldsymbol{\chi} = \boldsymbol{\varepsilon} + p\,\mathbf{v}$	total enthalphy or chi in dyne centimeters
$\chi = \epsilon + p\,v$	enthalpy per unit mass in dyne centimeters per gram.
$\boldsymbol{\psi} = \boldsymbol{\varepsilon} - \theta\,\mathbf{n}$	total psi in dyne centimeters
$\psi = \epsilon - \theta\,\eta$	psi per unit mass in dyne centimeters per gram.
$\mathbf{W}$ and W	total work and work per unit mass respectively, received by the system in dyne centimeters and dyne centimeters per gram.
$\mathbf{Q}$ and Q	total heat and heat per unit mass respectively, received by the system in dyne centimeters and dyne centimeters per gram.

c_p	heat capacity per unit mass at constant pressure and concentration.
$l_p = -\theta\left(\frac{\partial v}{\partial t}\right)_{p, m_1, \ldots, m_n}$	latent heat of change of pressure per unit mass at constant temperature and concentration.
$c_v = c_p + l_p\left(\frac{\partial p}{\partial t}\right)_{v, m_1, \ldots, m_n}$	heat capacity per unit mass at constant volume and concentration.
$l_v = l_p\left(\frac{\partial p}{\partial v}\right)_{t, m_1, \ldots, m_n}$	latent heat of change of volume per unit mass at constant temperature and concentration.
l_{m_k}	"reversible" heat of change of mass of component k where temperature, pressure, and the other component masses are constant.
$l_{w_k} = l_{m_k} + l_p\left(\frac{\partial p}{\partial m_k}\right)_{t, v, m_i}$	"reversible" heat of change of mass of component k where temperature, volume, and the other component masses are constant.
μ_k	"chemical potential" of component k in the phase.
s	a parameter; used as curve or path, or time.
ρ	density
E	electromotive force
q	quantity of electricity
c	velocity of light
g	acceleration of gravity
(x, y, z), (ξ, η, ζ)	coordinates
u, v, w	projections of displacement vectors on the x, y, z axes respectively.
$e_1, \ldots, e_6$	strain components. (see sections 53, 60)
X, Y, Z	projections of body force vectors on the x, y, z, axes respectively
$X_1 = X_x$ $X_2 = Y_y$ $X_3 = Z_z$ $X_4 = Z_y = Y_z$ $X_5 = X_z = Z_x$ $X_6 = Y_x = X_y$	Stress components (*e. g.* Z_y = traction along the z-axis across the y-plane). Considered positive when they are tensions and negative when pressures. (section 65)
c_x	heat capacity per unit volume at constant stress.

c_e	heat capacity per unit volume at constant strain.
$l_{x_1} = \theta\left(\frac{\partial e_1}{\partial \theta}\right)_{X_1, \ldots, X_6}$	latent heat of change of stress parallel to the x-axis per unit volume where temperature and the other stresses are constant. Similarly for l_{x_i}, i = 2, . . ., 6
$l_{e_1} = \sum l_{x_k}\left(\frac{\partial X_k}{\partial e_1}\right)_{t, e_2, \ldots, e_6}$	Latent heat of change of strain along the x-axis per unit volume where temperature and the other strains are constant. Similarly for l_{e_i}, i = 2, . . ., 6
c_{ih}, (i, h) 1, . . ., 6	"Elastic constants" (section 81).
K	Bulk modulus or modulus of compression
$\beta = \frac{1}{K}$	compressibility
E	Young's modulus
σ	Poisson's ratio
R	modulus of rigidity
Δ	cubical dilatation (section 86)
$\epsilon_t\ (t, p, m_1)$	is a notation for $\left(\frac{\partial \epsilon}{\partial t}\right)_{p, m_1}$

Tables I and II—$\mathbf{m}_a$ denotes all the component masses, *i.e.* $\mathbf{m}_1, \cdots, \mathbf{m}_n$; $\mathbf{m}_i$ all except $\mathbf{m}_k$; $\mathbf{m}_j$ all except $\mathbf{m}_h$; $\mathbf{m}_g$ all except $\mathbf{m}_k$ and $\mathbf{m}_h$; $\mathbf{m}_b$ all except $\mathbf{m}_k$, $\mathbf{m}_h$ and $\mathbf{m}_y$.

Table II—Subscripts: $\mathbf{e} = \varepsilon$; $\mathbf{n} = \mathfrak{n}$; $\mathbf{x} = \chi$; $\mathbf{y} = \psi$; $\mathbf{z} = \zeta$.

THERMODYNAMIC RELATIONS IN MULTI-COMPONENT SYSTEMS

By Roy W. Goranson

PART I

CHAPTER I

Fundamental Ideas

UNDEFINED CONCEPTS OR DIRECTLY MEASURABLE QUANTITIES

Let us begin our inquiry by considering what the undefined concepts or directly measurable quantities of thermodynamics are. We take them to be: length, time, mass, force, and temperature.

The point of view adopted here in considering the physical concepts is that of Bridgman.[1] In treating these physical concepts there is no intent here to make the investigation an exhaustive one since that is not possible in a work of this size. However, the analysis of any theory must begin with such a treatment since these concepts are the starting point and hence an integral part of the theory.

1. Length: Bridgman says: "Our task is to find the operations by which we measure the length of any concrete physical object. We begin with objects of our commonest experience, such as a house or a house lot. What we do is sufficiently indicated by the following rough description. We start with a measuring rod, lay it on the object so that one of its ends coincides with one end of the object, mark on the object the position of the other end of the rod, then move the rod along in a straight line extension of its previous position until the first end coincides with the previous position of the second end, repeat this process as often as we can, and call the length the total number of times the rod was applied. This procedure, apparently so simple, is in practice exceedingly complicated, and doubtless a full description of all the precautions that must be taken would fill a large treatise. We must, for example, be sure that the temperature of the rod is the standard temperature at which its length is defined, or else we must make a

[1] P. W. Bridgman, The Logic of Modern Physics (Macmillan, New York), 1927.

correction for it; or we must correct for the gravitational distortion of the rod if we measure a vertical length; or we must be sure that the rod is not a magnet or is not subject to electrical forces.

. . . Practically of course precautions such as these are not mentioned, but the justification is in our experience that variations or procedure of this kind are without effect on the final result. But we always have to recognize that all our experience is subject to error, and that at some time in the future we may have to specify more carefully the acceleration, for example, of the rod in moving from one position to another if experimental accuracy should be so increased as to show a measurable effect. In principle the operations by which length is measured should be uniquely specified. . ."

* * *

"We . . . are also compelled to modify our procedures when we go to small distances. Down to the scale of microscopic dimensions a fairly straightforward extension of the ordinary measuring procedure is sufficient, as when we measure a length in a micrometer eyepiece of a microscope. This is of course a combination of tactual and optical measurements, and certain assumptions, justified as far as possible by experience, have to be made about the behavior of light beams. These assumptions are of a quite different character from those which give us concern on the astronomical scale, because here we meet difficulty from interference effects due to the finite scale of the structure of light, and are not concerned with a possible curvature of light beams in the long reaches of space. Apart from the matter of convenience, we might also measure small distances by the tactual method.

"As the dimensions become smaller, certain difficulties become increasingly important that were negligible on a larger scale. In carrying out physically the operations equivalent to our concepts, there are a host of practical precautions to be taken which could be explicitly enumerated with difficulty, but of which nevertheless any practical physicist is conscious. Suppose, for example, we measure length tactually by a combination of Johanssen gauges. In piling these together, we must be sure that they are clean, and

are thus in actual contact. Particles of mechanical dirt first engage our attention. Then as we go to smaller dimensions we perhaps have to pay attention to adsorbed films of moisture, then at still smaller dimensions to adsorbed films of gas, until finally we have to work in a vacuum, which must be the more nearly complete the smaller the dimensions. About the time that we discover the necessity for a complete vacuum, we discover that the gauges themselves are atomic in structure, that they have no definite boundaries, and therefore no definite length, but the length is a hazy thing, varying rapidly in time between certain limits. We treat this situation as best we can by taking a time average of the apparent positions of the boundaries, assuming that along with the decrease of dimensions we have acquired a corresponding extravagant increase in nimbleness. But as the dimensions get smaller continually, the difficulties due to this haziness increase indefinitely in percentage effect, and we are eventually driven to give up altogether. We have made the discovery that there are essential physical limitations to the operations which defined the concept of length. . . . At the same time that we have come to the end of our rope with our Johanssen gauge procedure, our companion with the microscope has been encountering difficulties due to the finite wave length of light; this difficulty he has been able to minimize by using light of progressively shorter wave lengths, but he has eventually had to stop on reaching X-rays. Of course this optical procedure with the microscope is more convenient and is therefore adopted in practice."

When using more than one operation we must so choose them that they give, within the experimental error, the same numerical results in the domain in which the two sets of operations may be both applied; but we must recognize in principle that in changing the operations we have really changed the concept and that to use the same name for these different concepts over the entire range is dictated only by considerations of convenience. We must therefore always be prepared some day to find that an increase in experimental accuracy may show that the two sets of operations which give the same results in the more ordinary part of the domain of experience lead to measurably different results in the more

unfamiliar parts of the domain and thus must keep aware of the joints in our conceptual structure if we hope to render unnecessary the services of the unborn Einsteins.

We choose as our concept the tactual and assume that the optical operation gives identical results. This is verified by our past experience but must remain subject to the possible modification stated above.

The concept of volume is an extension of that of length. The concept arises from the fundamental sensations in which all geometrical notions are founded. If we are in a room the presence of rigid bodies, as furniture, may prevent us from moving our limbs in certain ways and as a natural consequence we might say that the rigid bodies which prevent certain motions occupy space corresponding to the motions which they render impossible and which may be called motor space.[1] Further, before the establishing of a quantitative method of measurement, we conclude from our vague senses that the amount of muscular motion excluded by a body indicates its volume.

To obtain the general notion of a difference between different positions in space that does not depend upon their occupation by one material system rather than another and to find means of measuring magnitudes to replace the vague and subjective muscular sensations, a system of coordinates must be introduced.

We accept as a fact that to every position in space defined by muscular motion there corresponds (if the system of coordinates is rightly chosen) one and only one set of coordinates.

2. Time: All ideas of time depend on the immediate judgments of "before," "after," and "simultaneous with" at the place of the observer.

These relations are such that events can be arranged in a numerical order and thus physical judgments can be rendered in respect of time.

The measurement of "time" is effected by means of periods which are properties of systems or individual bodies. The reali-

[1] Poincaré says if there were no solid bodies in nature there would be no geometry.

zation of a standard series is obtained by the use of isoperiodic systems, namely such as are characterized by a series of events A, B, C, D, which may be as long as we please, the periods AB, BC, CD, . . . being all equal as tested against some other period. Accordingly in such systems the periods AB, AC, AD provide the integral members of the standard series. The fractional members are provided by other isoperiodic systems, one event of these periods being made simultaneous with a member of the integral series. Pendulums and clocks are examples of isoperiodic systems. (For our limited purposes the ultimate isoperiodic system providing an unending series of events is the rotating Earth.)

With the standard system of periods established, it is then possible to measure magnitudes, called time-intervals.

The concept of time is determined by the operations by which it is measured, but the physical operations at the basis of the measurement of time have never been subjected to the critical examination which seems to be required. We might seek to specify the measurement of time in purely mechanical terms, as for instance in terms of the vibration of a tuning fork, or the rotation of a flywheel. But here we encounter difficulties. As Bridgman says, "We want to use the clock as a physical instrument in determining the laws of mechanics, which of course are not determined until we can measure time, and we find that the laws of mechanics enter into the operation of the clock.

"The dilemma which confronts us here is not an impossible one, and is in fact of the same nature as that which confronted the first physicist who had to discover simultaneously the approximate laws of mechanics and geometry with a string which stretched when he pulled it. We must first guess at what the laws are approximately, then design an experiment so that, in accordance with this guess, the effect of motion on some phenomenon is much greater than the expected effect on the clock, then from measurements with uncorrected clock time find an approximate expression for the effect of motion on mass or length, with which we correct the clock, and so on ad infinitum. However, so far as I know, the possibility of such a procedure has not been analyzed, and until the analysis is given, our complacency is troubled by a real dis-

quietude, the intensity of which depends on the natural skepticism of our temperament.

* * *

"This discussion of the concept of time will doubtless be felt by some to be superficial in that it makes no mention of the *properties* of the physical time to which the concept is designed to apply. For instance, we do not discuss the one dimensional flow of time, or the irrevocability of the past. Such a discussion, however, is beyond our present purpose, and would take us deeper than I feel competent to go, and perhaps beyond the verge of meaning itself. Our discussion here is from the point of view of operations: we assume the operations to be given, and do not attempt to ask why precisely these operations were chosen, or whether others might not be more suitable. Such properties of time as its irrevocability are implicitly contained in the operations themselves, and the physical essence of time is buried in that long physical experience that taught us what operations are adapted to describing and correlating nature. We may digress, however, to consider one question. It is quite common to talk about a reversal of the direction of flow of time. Particularly, for example, in discussing the equations of mechanics, it is shown that if the direction of flow of time is reversed, the whole history of the system is retraced. The statement is sometimes added that such a reversal is actually impossible, because it is one of the properties of physical time to flow always forward. If this last statement is subjected to an operational analysis, I believe that it will be found not to be a statement about nature at all, but merely a statement about operations. It is *meaningless* to talk about time moving backward: by definition, *forward* is the direction in which time flows."[1]

3. Mass: The notion of mass is also sufficiently familiar, provided we take it just as it comes, in its naïve original form.

We may try to convey the idea by saying that mass is the quantity of matter which a body contains, but this is not a definition of mass; it is merely a statement of the concept in different words.

[1] P. W. Bridgman, The Logic of Modern Physics, pp. 71, 78-79.

We agree that the unit of mass shall be the mass of a standard platinum-iridium cylinder known as the International Prototype kilogram. Measuring the mass of a body consists in comparing its mass with that of the standard cylinder. In order that this might be done conveniently, it was first necessary to construct bodies of the same mass as that of the standard cylinder, and then to make a whole series of bodies whose masses were 1/2, 1/10, 1/100, 1/1000, etc., of the standard mass; in other words to construct a set of standard masses. This we are able to do quite easily since the mass of a compound body formed by uniting two or more bodies is, by the nature of the concept, equal to the sum of the masses of the separate bodies. Thus two equal masses which are together equal to the standard unit mass will each be one-half the mass of the unit; four equal masses which together are equal to the standard unit mass will each be one-quarter of the mass of the unit; and so on.

With the aid of such a set of standard masses the determination of the mass of any unknown body is made by first placing the body upon one pan of a balance and counterpoising with shot, paper, etc., and then replacing the unknown body by as many of the standard masses as are required to bring the pointer back to zero again. The mass of the body is equal to the sum of these standard masses.[1]

We have assumed that the gravitational field of the earth acts equally on equal masses. For our purposes this method of determination of mass is sufficient. If we, however, go to high velocities and to extended space we are in difficulties since we do not know whether the force of gravity is independent of velocity at high velocities. Here the concepts of force and mass lose their definiteness and become partially fused since there are no operations by which force can be obtained as a function of velocity without knowing the mass or any operation by which mass can be measured without knowing the force.

[1] This method of determination of mass is called the method of substitution. See R. A. Millikan and H. G. Gale, A First Course in Physics, 1913, p. 6.

4. Force: The idea of force is also familiar in the form of a push or pull.[1] To find how hard we are pulling when we hold a kite string it is only necessary to tie a spring balance to the end of the string and note how far the spring is stretched.

The unit of force is perfectly arbitrary, being simply the force which causes any agreed-upon amount of stretch in a standard spring; and a complete scale of multiples and submultiples of the unit is readily established by simply opposing one or more unmarked springs, in various combinations, against the standard, or unit, spring, and marking the positions reached by the pointers.[2]

By means of a portable, graduated spring balance thus constructed we are able theoretically to measure forces for systems in equilibrium in terms of the arbitrarily chosen unit of force represented by our originally chosen standard spring. "We next extend the force concept to systems not in equilibrium, in which there are accelerations, and we must conceive that at first all our experiments are made in an isolated laboratory far out in empty space, where there is no gravitational field. We here encounter a new concept, that of mass, which as it is originally met is entangled with the force concept, but may later be disentangled by a process of successive approximations. The details of the various steps in the process of approximation are very instructive as typical of all methods in physics, but need not be elaborated here. Suffice it to say that we are eventually able to give to each rigid material body a numerical tag characteristic of the body, such that the product of this number and the acceleration it receives under the action of any given force applied to it by a spring balance is numerically equal to the force, the force being defined, except for a correction, in terms of the deformation of the balance, exactly as it was in the static case. In particular, the relation found between mass, force, and acceleration applies to the spring balance itself by which the force is applied, so that a correction has to be applied for a diminution of the force exerted by the balance arising from its own acceleration.

[1] W. F. Osgood, Introduction to the Calculus, 1922, p. 348.

[2] This process involves no knowledge of Hooke's Law; it merely assumes that the elastic properties of the spring do not vary with the time.

"We now extend the scope of our measurements by bringing our laboratory into the gravitational field of the earth, and immediately our experience is extended, in that we continually see bodies accelerated with no spring balance (that is, no force) acting on them. We extend the concept of force, and say that any body accelerated is acted on by a force, and the magnitude of this force is defined as that which would have been necessary to produce in the same body the same acceleration with a spring balance in empty space. There is physical justification for this extension in that we find we can remove the acceleration which a body acquires in a gravitational field by exerting on it with a spring balance a force of exactly the specified amount in the opposite direction."[1] In extending the notion of force to systems not in equilibrium (moving in force fields) we have changed the character of the concept—the force acting on the body is now measured in terms of mass-acceleration.

The hypothesis is made that these two operations measure the same thing.

5. Temperature: The concept "which sets thermodynamics off apart from the simple subjects is probably that of temperature. In origin this concept was without question physiological in much the same way as the mechanical concept of force was physiological. But just as the force concept was made more precise, so the temperature concept may be more or less divorced from its crude significance in terms of immediate sensation and be given a more precise meaning. This precision may be obtained through the notion of equilibrium states."[2]

The idea of temperature is obtained from our sensation of how hot or how cold a body is. By touching a number of bodies with the hand, we can arrange them roughly in order, from the coldest to the hottest. Such an order, however, is likely to be modified by a second series of observations. By touching the bodies with a glass tube enlarged at the bottom and partly filled with mercury, and determining the place of any body in the series according to

[1] P. W. Bridgman, The Logic of Modern Physics, pp. 102-103.
[2] P. W. Bridgman, *op. cit.*, pp. 117, 118.

the amount by which the level of the mercury rises when the tube is placed in contact with it, we secure an order which is modified only in the cases of a few bodies when a second series of observations is made. If, finally, we use an apparatus which we call a hydrogen gas thermometer, we can repeat the observations many times for all the bodies, each of which is kept under constant conditions, without finding it necessary to change the order. The reading of this hydrogen gas thermometer, when placed in contact with a body, we agree to take as the temperature of the body.

This will be true only if the mass of the thermometer is infinitely small compared with the masses of the bodies whose temperatures we are measuring. We find that the temperature reading will depend on the relative masses of the systems measured and the thermometer, and on the kind of material used as container for the hydrogen gas. Thus, since in actual practice the thermometer must be of finite size with respect to the system measured, suitable corrections must be made for the mass of the thermometer and kind of material used in its construction.

By means of this thermometer we can measure temperature just as we can measure force by means of the spring balance. Our concept is that the reading of this thermometer corresponds to a physical quantity.[1]

6. General Note regarding the physical quantities: Among the reasons which justify the introduction of the quantities enumerated as physical quantities is the empirical observation of certain rather obvious relations.

In the case of temperature the first of these relations is as follows: Let A, B and C be three bodies. If A undergoes no change of

[1] For purposes of thermodynamics as we have limited them temperature can certainly not be defined in terms of flow of heat. Thus the statement by G. N. Lewis and M. Randall on page 57 of "Thermodynamics and the free Energy of Chemical Substances," for example, that "if there can be no thermal flow from one body to another, the two bodies are at the same temperature; but if one can lose energy to the other by thermal flow, the temperature of the former is the greater" would be incorrect as a definition of temperature.

On this subject P. W. Bridgman, Logic of Modern Physics, p. 125, says: "The essential fact that a quantity of heat can by itself be defined only in terms of a drop of temperature is somewhat obscured by the usual method of thermodynamic analysis."

volume on being placed in contact with B and if further B undergoes no change of volume on being placed in contact with C, then it is a matter of experience that A will undergo no change of volume on being placed in contact with C. (Thus if A and B are in thermal equilibrium, and B and C are in thermal equilibrium, then A and C are in thermal equilibrium.) The other relation is: If the volume of A decreases and that of B increases when they are placed in contact, if further the volume of B decreases and the volume of C increases when they are placed in contact, then we usually find that the volume of A will decrease and that of C will increase if they are placed in contact. (Thus if the temperature of A is greater than that of B and the temperature of B is greater than that of C then the temperature of A is greater than that of C.)

In the case of force analogous relations are observed. Let A, B and C be three spring balances each with a pointer attached to the spring. If A is extended until its pointer is opposite an arbitrary notch on its frame when opposed by B and a notch is then cut opposite the pointer of B on its frame, if further B is extended until its pointer is opposite the notch on its frame when opposed by C and a notch is then cut opposite the pointer of C on its frame, then it is found that if A is extended until its pointer is opposite the notch on its frame when opposed by C the pointer of C will be opposite the notch on its frame.

A statement regarding inequalities analogous to that made in the case of temperature could also be made in the case of force.

Similarly for the other physical quantities.

These, then, are our fundamental quantities: length, as measured by a meter bar; time, by a chronometer; mass, by a balance; force, by a spring balance; and temperature, as measured by a hydrogen gas thermometer.

Thermodynamics, like the other branches of physics, deals with two kinds of quantities: those directly measurable experimentally[1] and those which are defined in terms of the directly measurable quantities by means of mathematical equations. In addition to the equations of definition of the quantities not directly measurable two other kinds of relations are present in the subject. We assume

[1] Called undefined concepts.

certain equations between various quantities of the two classes to be true as physical hypotheses: our knowledge of the truth of these equations may be directly or indirectly obtained from experiment; they can not, however, be derived mathematically from any relations previously obtained by definition and physical hypothesis. Lastly we have equations derived mathematically from equations already obtained by definition or physical hypothesis. According to Bridgman equations derived mathematically are never definitely known to be true physically until they are proved so experimentally. Poincaré reaches the same conclusion and adds that the function of mathematics is not to produce new truth from old physical truth but to suggest new significant experiments. Moreover the mathematically derived consequences of a physical hypothesis may be useful in testing the hypothesis in case the hypothesis can not be verified directly by experiment. If the mathematically derived consequences are found to be true experimentally we conclude that the probability that the hypothesis is also true is thereby increased.[1]

The directly measurable quantities were first known qualitatively and roughly quantitatively by physical methods. They are accurately defined qualitatively and quantitatively, as stated by Bridgman, in terms of physical operations.

7. Property:[2] By stating the properties of a system we describe its condition at a given instant. When a piece of steel is subjected to mechanical treatment its final volume is a property of the steel at the end of the process. These properties are either length, force, temperature, or mass, or quantities expressed in terms of the fundamental four quantities.

Properties can be divided into two classes. The mass or volume of two identical systems, say two kilogram weights of brass or two exactly similar balloons of hydrogen, is double that of each one.

[1] F. C. S. Schiller (Studies in the History and Method of Science, Edited by Charles Singer, Clarendon Press, 1917, p. 268) states that the logicians have pointed out for 2000 years that this involves a breach of formal logic. Schiller agrees that it does but concludes not that this fundamental method of science is wrong but rather that formal logic is worse than useless in science.

[2] G. N. Lewis and M. Randall, Thermodynamics, 1923, pp. 12, 13.

Such properties are called extensive. The temperature of the two identical objects, on the other hand, is the same as that of either one. Properties of this kind are called intensive. Intensive properties are in many cases derived from the extensive properties. Thus while mass and volume are both extensive, the density and specific volume are intensive properties.

STATE[1]

In thermodynamic considerations we say that the state of a system is given when all of its intensive properties are fixed.

In thermodynamics we first consider systems in which the properties vary continuously from point to point.

A system, the properties of which are the same at all points, we define, following Gibbs,[2] as a homogeneous system.

A heterogeneous system is defined as one consisting of two or more homogeneous parts.

EQUILIBRIUM STATE

If the properties of a system undergo no change after the lapse of a period of time, no matter how greatly it is extended, the system is said to be in a state of equilibrium.

Properties here refer to the external conditions of the system. If we were to examine the internal conditions of, say, a gas on a small enough scale we would then observe inhomogeneities resulting from Brownian movement and other phenomena. All that we are concerned with, however, is that, for an equilibrium state, all such interactions in the system will be subject to the condition that the external properties remain constant as measured by our scale of operations.

Our first physical hypothesis is that a homogeneous system is in an equilibrium state.[3]

[1] G. N. Lewis and M. Randall, op. cit., p. 12.

[2] The Collected Works of J. Willard Gibbs, Vol. 1, p. 63.

[3] The assumption that one of the conditions for the state of a homogeneous system to be a state of equilibrium is that the pressure be the same at all points is identical with one of the postulates of statics.

CHAPTER II

Simple homogeneous systems

UNIT MASS SYSTEMS

In thermodynamics we consider first simple homogeneous systems in which the composition, density, and temperature are the same at all points, and which are subject to a pressure which is the same at all points and which is the same in all directions at any given point.

8. Equation of State or characteristic equation: The specific volume of a system of this type is defined as the total volume divided by the total mass,

$$v = \frac{\mathrm{V}}{\mathrm{m}}$$

We assume as a physical hypothesis that when any two of the properties pressure, specific volume, and temperature of a particular system are given, the third is determined, or that

$$\varphi\,(p, v, t) = 0,$$

where p denotes the pressure in dynes per square centimeter, v the specific volume in cubic centimeters per gram, and t the temperature in centigrade degrees on the scale of the hydrogen gas thermometer.

Each pair of values of the two properties which can be varied independently may be represented by a point in a Cartesian coordinate plane, the abscissa and ordinate of which are proportional to the values of the properties. Thus we may "say" that each point represents a state of the system.

A series of states in which the properties vary continuously may be spoken of as a continuous series of states and may be represented geometrically by a continuous curve in the coordinate plane. A hypothetical "reversible process" of expansion is then a

continuous series of equilibrium states which can be represented by a continuous curve in the (t, p) coordinate plane or analytically by the equation $t = f(p)$. Following the usage of Gibbs,[1] "For the sake of brevity it will be convenient to use language which attributes to the diagram properties which belong to the associated states of the body. Thus it can give rise to no ambiguity, if we speak of the volume or temperature of a point in the diagram instead of the volume or temperature of the body in the state associated with the point. . . . In like manner also we may speak of the body moving along a line in the diagram, instead of passing through the series of states represented by the line."

9. Definitions of work and heat: Let us take as the two independent variables t and p, and let us represent t by the ordinate and p by the abscissa of a point in a Cartesian coordinate plane. Then the work received by the system during the "reversible" expansion, *i.e.* in passing through a series of states represented by points of the continuous curve or simply work of the path,[2] W, in ergs per gram is defined geometrically as the area of the cylinder erected on the curve or analytically as the line integral,[3]

$$-\int_{t_0, p_0}^{t, p} p \frac{\partial v}{\partial t} dt + p \frac{\partial v}{\partial p} dp \qquad (1)$$

$$W = -\int_{t_0, p_0}^{t, p} p \frac{\partial v}{\partial t} dt + p \frac{\partial v}{\partial p} dp \qquad (2)$$

[1] J. Willard Gibbs, Collected Works, Vol. 1, footnote, p. 3.

[2] The terms "work of the path" and "heat of the path" are adopted from Gibbs, Scientific Papers, Vol. 1 (1906), p. 3. He says "W and H (= our Q) are not functions of the state of the body (or functions of any of the quantities, v, p, t, ϵ and η), but are determined by the whole series of states through which the body is supposed to pass . . . Suppose the body to change its state, the points associated with the states through which the body passes will form a line, which we may call the *path* of the body. The conception of a path must include the idea of direction, to express the order in which the body passes through the series of states."

[3] For a definition of line integral see W. F. Osgood, Lehrbuch der Funktionentheorie, p. 125.

and the heat received or heat of the path, Q, in ergs per gram is defined as the line integral,

$$\int_{t_0,\, p_c}^{t,\, p} l_p(t, p)\, dp + c_p(t, p)\, dt, \tag{3}$$

$$Q = \int_{t_0,\, p_0}^{t,\, p} c_p(t, p)\, dt + l_p(t, p)\, dp \tag{4}$$

where c_p and l_p are some continuous functions of Δt and Δp.

10. Reversible *vs.* irreversible processes: A system can not actually pass through a continuous series of equilibrium states, since according to the hypothesis made by all authors an equilibrium state is one in which the properties of the system do not change with time.[1]

In reality the consideration of the "irreversible" processes, *i.e.* continuous series of non-equilibrium states, must therefore logically precede the consideration of "reversible" processes.[2]

The quantity customarily called the heat received in the reversible process, but correctly named by the phrase of Gibbs "work or heat of a line" can in reality only be evaluated as the limit of the quadruple integral which is measured physically and which is the heat received in an irreversible process.

Following the customary methods in treating mathematical and physical subjects, we shall, however, first take up the simpler part of the subject although it must logically be preceded by the more general case which is more complex and therefore is left until the end.

[1] See § 91

[2] J. Willard Gibbs, Collected Works, Vol. 1, p. 55. Gibbs has said: "As the difference of the values of the energy for any two states represents the combined amount of work and heat received or yielded by the system when it is brought from one state to the other, and the difference of entropy is the *limit* (my italics) of all the possible values of the integral $\int \frac{dQ}{t}$, (dQ denoting the element of the heat received from external sources, and t the temperature of the part of the system receiving it), the varying values of the energy and entropy characterize in all that is essential the effects producible by the system in passing from one state to another."

There is no objection to doing this provided that in the portion left over for the time being we do not make use of any of the theorems brought out in the part treated first. In fact, the part left over is really so treated that it is logically first and anyone desiring a logical treatment may secure this by reading it first.

It may also be pointed out that the treatment of so-called reversible processes does not make use of theorems deduced in the treatment of irreversible processes, and thus the mathematical treatment of each of these two parts is complete in itself. The necessity for the logical precedence of the irreversible processes is due to the physical, not the mathematical, nature of the situation.

For the convenience of those readers who do not wish to spend time going through the treatment of irreversible processes the customary language of reversible processes, which Gibbs also has made use of (although as has been pointed out above he clearly recognized that a "reversible" process in reality must be treated as the limit of "irreversible" processes), will be used in the first part, and thus we shall speak of the heat received and work done in reversible processes.

11. Transformation of heat and work integrals: Now by hypothesis p can be expressed as a function of t and v,

$$p = \mathrm{f}_1\,(t, v) \tag{1}$$

Thus

$$\int_{t_0,\,p_0}^{t,\,p} c_p\,\mathrm{d}t + l_p\,\mathrm{d}p = \int_{t_0,\,v_0}^{t,\,v} \left[c_p + l_p \frac{\partial p}{\partial t}\right] \mathrm{d}t + l_p \frac{\partial p}{\partial v}\,\mathrm{d}v \tag{2}$$

Let us define

$$c_p + l_p \frac{\partial p}{\partial t} = c_v \tag{3}$$

and

$$l_p \frac{\partial p}{\partial v} = l_v, \tag{4}$$

then

$$\int_{t_0,\,p_0}^{t,\,p} c_p\,\mathrm{d}t + l_p\,\mathrm{d}p = \int_{t_0,\,v_0}^{t,\,v} c_v\,\mathrm{d}t + l_v\,\mathrm{d}v \tag{5}$$

Similarly for the work integral we can make the transformation,

$$-\int_{t_0,\, p_0}^{t,\, p} p\frac{\partial v}{\partial t}\,\mathrm{d}t + p\frac{\partial v}{\partial p}\,\mathrm{d}p = -\int_{t_0,\, v_0}^{t,\, v} p\,\mathrm{d}v \tag{6}$$

12. Differentials and derivatives of heat and work: Now the coordinates of the curve $t = \mathrm{f}\,(p)$ can be expressed as functions of the distance along the regular curve, s, measured from an arbitrary point,

$$t = \varphi\,(\mathrm{s}),\ p = \xi\,(\mathrm{s}),\ (\mathrm{l} \leqq \mathrm{s} \leqq \mathrm{L}),$$

and

$$\int_{t_0,\, p_0}^{t,\, p} c_p\,\mathrm{d}t + l_p\,\mathrm{d}p$$

is a notation for

$$\lim_{\mathrm{n} \to \infty} \sum_{\mathrm{k}=0}^{\mathrm{n}-1} c_p\,(t'_{\mathrm{k}}, p'_{\mathrm{k}})\,\Delta t_{\mathrm{k}} + \mathrm{l}_{\mathrm{p}}\,(t'_{\mathrm{k}}, p'_{\mathrm{k}})\,\Delta p_{\mathrm{k}},$$

where the interval of the regular curve from (t_0, p_0) to (t, p) is divided into n parts by the points $\mathrm{s}_0 = \mathrm{l}, \mathrm{s}_1, \mathrm{s}_2, \ldots\ldots\ldots\ldots, \mathrm{s}_{\mathrm{n}-1}$, $\mathrm{s}_{\mathrm{n}} = \mathrm{L}$, and $\mathrm{L} - \mathrm{l}$ is the length of s. Δt_{k} and Δp_{k} denote the differences $t_{\mathrm{k}+1} - t_{\mathrm{k}}$ and $p_{\mathrm{k}+1} - p_{\mathrm{k}}$, and $(t'_{\mathrm{k}}, p'_{\mathrm{k}})$ is an arbitrary point of the kth arc, $(\mathrm{s}_{\mathrm{k}-1}, \mathrm{s}_{\mathrm{k}})$.

To evaluate the limit we may write the summand in the form:

$$\left[c_p\,(t'_{\mathrm{k}}, p'_{\mathrm{k}})\,\frac{\Delta t_{\mathrm{k}}}{\Delta \mathrm{s}} + \mathrm{l}_p\,(t'_{\mathrm{k}}, p'_{\mathrm{k}})\,\frac{\Delta p_{\mathrm{k}}}{\Delta \mathrm{s}}\right]\Delta \mathrm{s}.$$

Now since

$$\lim_{\Delta \mathrm{s} \to 0} \frac{\Delta t}{\Delta \mathrm{s}} = \cos\alpha,\quad \lim_{\Delta \mathrm{s} \to 0} \frac{\Delta p}{\Delta \mathrm{s}} = \cos\beta,$$

it follows, by Duhamel's theorem, that

$$\lim_{\mathrm{n} \to \infty} \sum_{\mathrm{k}=0}^{\mathrm{n}-1} \left[c_p\,(t'_{\mathrm{k}}, p'_{\mathrm{k}})\,\Delta t_{\mathrm{k}} + l_p\,(t'_{\mathrm{k}}, p'_{\mathrm{k}})\,\Delta p_{\mathrm{k}}\right] =$$

$$\lim_{n \to \infty} \sum_{k=0}^{n-1} \Big[c_p (t'_k, p'_k) \cos\alpha + l_p (t'_k, p'_k) \cos\beta \Big] \Delta s.$$

The limit of the latter sum is

$$\int_l^L \Big\{ c_p \Big[\varphi(s), \psi(s) \Big] \varphi'(s) + l_p \Big[\varphi(s), \psi(s) \Big] \psi'(s) \Big\} ds$$

$$= \int_l^L \Big[c_p (t, p) \cos\alpha + l_p (t, p) \cos\beta \Big] ds.$$

Any definite integral, as for example $\int_a^x f(t)\,dt$, where the limit a is regarded as fixed, is a function of the upper limit x, $F(x) = \int_a^x f(t)dt$, and the derivative of this function is $F'(x) = f(x)$. Hence for the integral

$$Q(L) = \int_l^L \Big\{ c_p \Big[\varphi(s), \psi(s) \Big] \varphi'(s) + l_p \Big[\varphi(s), \psi(s) \Big] \psi'(s) \Big\} ds$$

$$\frac{dQ(L)}{dL} = c_p \Big[\varphi(L), \psi(L) \Big] \frac{d\varphi(L)}{dL} + l_p \Big[\varphi(L), \psi(L) \Big] \frac{d\psi(L)}{dL}$$

Since L is the length of arc from an arbitrary point we can now replace it by s

$$\frac{dQ}{ds} = c_p \frac{dt}{ds} + l_p \frac{dp}{ds},$$

or

$$dQ = c_p\, dt + l_p\, dp,$$

where Δs is the independent variable and Δt and Δp depend upon Δs.

Similarly

$$dW = p\frac{\partial v}{\partial t}dt + p\frac{\partial v}{\partial p}dp$$

where Δs is the independent variable and Δt and Δp depend upon Δs.

13. Definitions of the heat capacities per unit mass: Let us suppose that t and p are the properties which can be varied independently.

Along any definite path s

$$p = k(s),\ t = l(s), \tag{1}$$

and hence

$$\frac{dQ}{ds} = c_p\frac{dt}{ds} + l_p\frac{dp}{ds} \tag{2}$$

and

$$\frac{dW}{ds} = -p\frac{\partial v}{\partial t}\frac{dt}{ds} - p\frac{\partial v}{\partial p}\frac{dp}{ds}. \tag{3}$$

In particular the path may be defined by the equations $t = s$, $p = K$, where K denotes a constant.

Along this path $\frac{dQ}{ds} = \frac{dQ}{dt}$ which we define as the heat capacity per unit mass at constant pressure.

Then

$$\frac{dQ}{dt} = c_p \tag{4}$$

and along this path

$$\frac{dW}{ds} = \frac{dW}{dt} = -p\frac{\partial v}{\partial t}. \tag{5}$$

Similarly, the path may be defined by the equations $t = K$, $p = s$, where K denotes a constant. Along this path $\frac{dQ}{ds} = \frac{dQ}{dp}$ which we define as the latent heat of change of pressure per unit mass at constant temperature.

Then

$$\frac{\mathrm{d}Q}{\mathrm{d}p} = l_p \tag{6}$$

and along this path

$$\frac{\mathrm{d}W}{\mathrm{ds}} = \frac{\mathrm{d}W}{\mathrm{d}p} = -\,p\frac{\partial v}{\partial p}. \tag{7}$$

Let us now suppose that t and v are the properties which can be varied independently.
Along any definite path

$$v = \mathrm{j}\,(\mathrm{s}),\; t = \mathrm{h}\,(\mathrm{s}),$$

where s denotes the distance along the curve, and hence

$$\frac{\mathrm{d}Q}{\mathrm{ds}} = c_v\frac{\mathrm{d}t}{\mathrm{ds}} + l_v\frac{\partial v}{\partial \mathrm{s}} \tag{8}$$

and

$$\frac{\mathrm{d}W}{\mathrm{ds}} = -\,p\frac{\mathrm{d}v}{\mathrm{ds}}. \tag{9}$$

In particular the path may be defined by the equations $t = s$, $v = \mathrm{K}$, where K denotes a constant.
Along this path $\frac{\mathrm{d}Q}{\mathrm{ds}} = \frac{\mathrm{d}Q}{\mathrm{d}t}$ which we define as the heat capacity per unit mass at constant volume.
Then

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \mathrm{c}_v \tag{10}$$

and along this path

$$\frac{\mathrm{d}W}{\mathrm{ds}} = \frac{\mathrm{d}W}{\mathrm{d}t} = 0. \tag{11}$$

Similarly, the path may be defined by the equations $t = \mathrm{K}$, $v = \mathrm{s}$, where K denotes a constant. Along this path $\frac{\mathrm{d}Q}{\mathrm{ds}} = \frac{\mathrm{d}Q}{\mathrm{d}v}$ which we define as the latent heat of change of volume per unit mass at constant temperature.

Then

$$\frac{dQ}{dv} = l_v, \tag{12}$$

and along this path

$$\frac{dW}{ds} = \frac{dW}{dv} = -p. \tag{13}$$

14. The first law of thermodynamics: The FIRST LAW of thermodynamics for homogeneous systems of unit mass states that

$$\epsilon(t, p) - \epsilon(t_0, p_0) = \int_{t_0,\, p_0}^{t,\, p} \left[c_p - p\frac{\partial v}{\partial t}\right] dt + \left[l_p - p\frac{\partial v}{\partial p}\right] dp \tag{1}$$

the line integral being extended along any path connecting the coordinates (t_0, p_0) and (t, p), where $\epsilon(t, p)$ denotes a function of the temperature and pressure of the system. This function defined by the preceding equation is called the internal energy of the system at the temperature t and pressure p of the coordinates (t, p). We further complete the definition of this function by defining $\epsilon(t_0, p_0)$ as zero, $\epsilon(t_0, p_0) = 0$.[1]

15. Geometric interpretation of the first law: The geometric interpretation of the line integral

$$\int_{s_0}^{s} \left\{\left[c_p - p\frac{\partial v}{\partial t}\right]\frac{dt}{ds} + \left[l_p - p\frac{\partial v}{\partial p}\right]\frac{dp}{ds}\right\} ds$$

extended along a particular path c is this: Let a cylinder be constructed on c as a generatrix,[2] its elements being perpendicular to the (t, p)-plane, and let the values of the function

$$F(s) = \left[c_p - p\frac{\partial v}{\partial t}\right]\frac{dt}{ds} + \left[l_p - p\frac{\partial v}{\partial p}\right]\frac{dp}{ds}$$

be laid off along the elements of this cylinder. Then the area of the

[1] For justification of this definition see § 93.

[2] I use the terms cylinder and generatrix in their usual geometric meaning. See W. F. Osgood and W. C. Graustein, Plane and Solid Analytic Geometry (Macmillan Co.) 1922, p. 532.

cylinder bounded by this curve and the generatrix represents the line integral. The first law of thermodynamics is the assumption that the areas of the cylinders erected on any two curves c_1 and c_2 connecting the coordinates (t_0, p_0) and (t, p) are equal.

16. The necessary and sufficient conditions for a line integral, expressed as the energy integral, to be independent of the path[1]:

Let $c_p - p\dfrac{\partial v}{\partial t}$ and $l_p - p\dfrac{\partial v}{\partial p}$ be two continuous functions in a simply connected region S.[2] Let $\dfrac{\partial}{\partial p}\left[c_p - p\dfrac{\partial v}{\partial t}\right]$ and $\dfrac{\partial}{\partial t}\left[l_p - p\dfrac{\partial v}{\partial p}\right]$ exist and be continuous in S.

The line integral

$$\int_C \left[c_p - p\frac{\partial v}{\partial t}\right] \mathrm{d}t + \left[l_p - p\frac{\partial v}{\partial p}\right] \mathrm{d}p \tag{1}$$

extended over a curve C lying in S and made up of a finite number of smooth pieces of curve, *i.e.* over a regular curve, is then and only then independent of the path if over the whole interior of S

$$\frac{\partial}{\partial p}\left[c_p - p\frac{\partial v}{\partial t}\right] = \frac{\partial}{\partial t}\left[l_p - p\frac{\partial v}{\partial p}\right] \tag{2}$$

For a fixed initial point the integral is then given as a continuous function $\epsilon\,(t, p)$ of the coordinates (t, p) of the endpoint of C, the first derivatives of ϵ being continuous and given by the equations

$$\epsilon_t\,(t, p) = c_p - p\frac{\partial v}{\partial t},\ \epsilon_p\,(t, p) = l_p - p\frac{\partial v}{\partial p}. \tag{3}$$

We shall prove first the necessity of our condition.

[1] The proof given here can be readily extended up to and including the n-component variable mass case with n + 2 independent variables. The proof by Green's theorem, although shorter, makes use of a great deal of geometric intuition, and, furthermore, it can not be extended to the other cases we deal with where more than two independent variables are present. For a further treatment of line integrals see W. F. Osgood, Lehrbuch der Funktionentheorie (B. G. Teubner, Leipzig), 1912, pp. 123-145; and A. Hurwitz and R. Courant, Allgemeine Funktionentheorie und Elliptische Funktionen (Julius Springer, Berlin), 1925, pp. 267-268.

[2] By a simply connected region is meant a region such that no closed curve drawn in the region contains in its interior a boundary point of the region. All other regions are called multiply connected.

For this we shall assume that our line integral is independent of the path, then it is given as a function $\epsilon\,(t, p)$ of the endpoint alone. We are to prove that this function is differentiable and that equations (3) hold.

Now since the line integral is assumed independent of what path we choose between the limits of integration provided that the path be a regular curve lying inside the simply connected region in which the line integral is defined, we may suppose that the path of integration goes from (a, b) to (t, p), and then from (t, p) to $(t + \Delta\, t, p)$ along a line parallel to the t-axis, *i.e.* along which p = constant.

$$\begin{aligned}\epsilon\,(t + \Delta\, t, p) - \epsilon\,(t, p) = {} & \text{integral from (a, b) to } (t, p) + \\ & \text{integral from } (t, p) \text{ to } (t + \Delta t, \mathrm{p}) - \\ & \text{integral from (a, b) to } (t, p). \\ = {} & \int_{t,\,p}^{t + \Delta\, t,\, p} \left[c_p\,(t, p) - p\,\frac{\partial v\,(t, p)}{\partial t}\right] \mathrm{d}t\end{aligned}$$

Applying the law of the mean, we write

$$\epsilon\,(t + \Delta t, p) - \epsilon\,(t, p) = \Delta t \left[c_p\,(t + \alpha\,\Delta\, t, p) - p\,\frac{\partial v\,(t + \alpha'\,\Delta t, p)}{\partial t}\right]$$

where

$$0 < \alpha < 1,\; 0 < \alpha' < 1.$$

Taking the limit when $\Delta\, t$ approaches zero of

$$\frac{\epsilon\,(t + \Delta t, p) - \epsilon\,(t, p)}{\Delta t}$$

gives $\epsilon_t\,(t, p) = c_p - p\,\dfrac{\partial v}{\partial t}$. Similarly $\epsilon_p\,(t, p) = l_p - p\,\dfrac{\partial v}{\partial t}$ (3′)

But $\dfrac{\partial^2 \epsilon}{\partial p\, \partial t} = \dfrac{\partial}{\partial p}\left[c_p - p\,\dfrac{\partial v}{\partial t}\right]$, and

$$\frac{\partial^2 \epsilon}{\partial t\, \partial p} = \frac{\partial}{\partial t}\left[l_p - p\,\frac{\partial v}{\partial p}\right];$$

and since $\dfrac{\partial}{\partial p}\left[c_p - p\dfrac{\partial v}{\partial t}\right]$ and $\dfrac{\partial}{\partial t}\left[l_p - p\dfrac{\partial v}{\partial p}\right]$ exist and are continuous by hypothesis, we have

$$\frac{\partial^2 \epsilon}{\partial p\, \partial t} = \frac{\partial^2 \epsilon}{\partial t\, \partial p}$$

Therefore $\dfrac{\partial}{\partial p}\left[c_p - p\dfrac{\partial v}{\partial t}\right] = \dfrac{\partial}{\partial t}\left[l_p - p\dfrac{\partial v}{\partial p}\right]$ (2′)

is a necessary condition for the line integral to be independent of the path.

To prove that condition (2) is sufficient that the line integral (1) be independent of the path, *i.e.* zero for all closed paths, we shall set up a function $\epsilon\,(t, p)$ such that

$\epsilon_t\,(t, p) = c_p - p\dfrac{\partial v}{\partial t}$ and $\epsilon_p\,(t, p) = l_p - p\,\dfrac{\partial v}{\partial p}$. If such a function exists the line integral is independent of the path, for then

$$\int_c \left[c_p - p\frac{\partial v}{\partial t}\right] \mathrm{d}t + \left[l_p - p\frac{\partial v}{\partial p}\right] \mathrm{d}p$$

$$= \int_c \epsilon_t\,(t, p)\, \mathrm{d}t + \epsilon_p\,(t, p)\, \mathrm{d}p$$

$$= \int_c \mathrm{d}\epsilon\,(t, p)$$

$$= \int_{\sigma_1 = (\mathrm{a},\, \mathrm{b})}^{\sigma_2 = (t_1,\, p_1)} \frac{\mathrm{d}}{\mathrm{d}\sigma}\,\epsilon\,(t, p)\, \mathrm{d}\sigma,\ t = \varphi\,(\sigma),\ p = \psi\,(\sigma),\quad \sigma_1 \leqq \sigma \leqq \sigma_2$$

$$= \epsilon\Big[\varphi\,(\sigma),\, \psi\,(\sigma)\Big]\Bigg|_{\sigma_1 = (\mathrm{a},\, \mathrm{b})}^{\sigma_2 = (t_1,\, p_1)}$$

$$= \epsilon\,(t_1, p_1) - \epsilon\,(\mathrm{a}, \mathrm{b}) \text{ by definition.}$$

We are to prove the existence of this function.

We shall prove that the line integral is zero for all closed rectangular paths.[1]

We shall define a function $\epsilon\,(t, p)$ inside and on the boundary of the rectangle

$$a_1 \leqq t_1 \leqq a_2,\ b_1 \leqq p_1 \leqq b_2,$$

where Γ is the path, by the equation

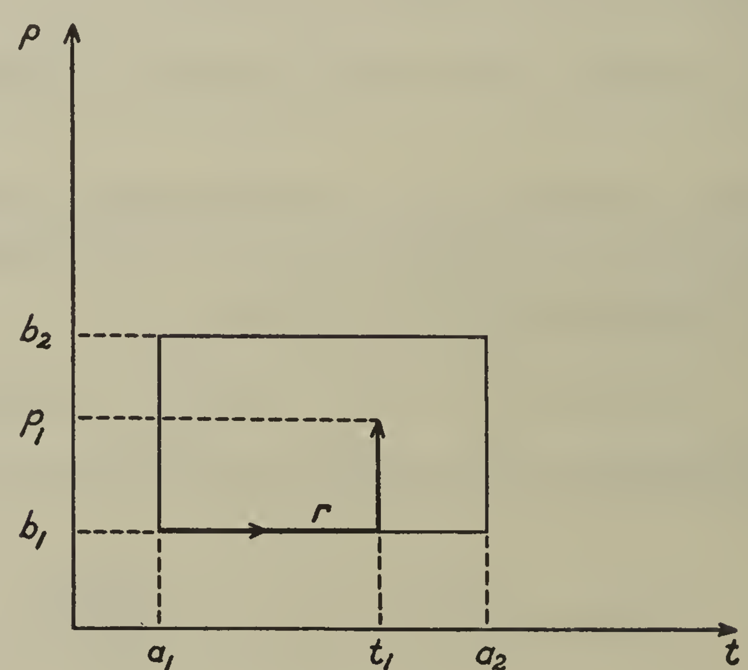

DIAGRAM 1

$$\epsilon\,(t_1, p_1) = \int_{\Gamma} \left[c_p - p\,\frac{\partial v}{\partial t}\right] dt + \left[l_p - p\,\frac{\partial v}{\partial p}\right] dp$$

$$= \int_{a_1}^{t_1} \left[c_p\,(t, b_1) - p\,\frac{\partial v\,(t, b_1)}{\partial t}\right] dt +$$

$$\int_{b_1}^{p_1} \left[l_p\,(t_1, p) - p\,\frac{\partial v\,(t_1, p)}{\partial p}\right] dp$$

[1] The assumption that the region S is a simply connected region is essential in our proof. From this assumption we make use of the fact that the integral extended over a simple closed stepped path can be represented as the sum of integrals over rectangles. Since this assumption checks the physical hypothesis we need not concern ourselves here with more complex regions.

Thus

$$\frac{\partial \epsilon\,(t_1, p_1)}{\partial t_1} = \left[c_p\,(t_1, \mathrm{b}_1) - p\,\frac{\partial v\,(t_1, \mathrm{b}_1)}{\partial t}\right] + \int_{\mathrm{b}_1}^{p_1} \frac{\partial}{\partial t_1}\left[l_p\,(t_1, p) - p\,\frac{\partial v\,(t_1, p)}{\partial p}\right] \mathrm{d}p$$

$$= c_p\,(t_1, \mathrm{b}_1) - p\,\frac{\partial v\,(t_1, \mathrm{b}_1)}{\partial t} + \int_{\mathrm{b}_1}^{p_1} \frac{\partial\left[c_p\,(t_1, p) - p\,\dfrac{\partial v\,(t_1, p)}{\partial t}\right]}{\partial p}\,\mathrm{d}p$$

$$= c_p\,(t_1, p_1) - p\,\frac{\partial v\,(t_1, p_1)}{\partial t}$$

Similarly

$$\frac{\partial \epsilon\,(t_1, p_1)}{\partial p_1} = l_p\,(t_1, p_1) - p\,\frac{\partial v\,(t_1, p_1)}{\partial p}$$

Therefore $\epsilon\ (t, p)$ is a function which satisfies equations (3) inside and on the boundary of the rectangle.

Therefore the line integral if it is extended over a path that does not leave the closed rectangle from $(t_0 = \mathrm{a}, p_0 = \mathrm{b})$ to $(t = t, p = p)$ is independent of the path; extended over the boundary of the rectangle the integral has the value zero.

Thus since

$$\frac{\partial}{\partial p}\left[c_p - p\,\frac{\partial v}{\partial t}\right] = \frac{\partial}{\partial t}\left[l_p - p\,\frac{\partial v}{\partial p}\right] \tag{2}$$

is a necessary and sufficient condition for the value of the line integral

$$\int_{t_0,\,p_0}^{t,\,p} \left[c_p - p\,\frac{\partial v}{\partial t}\right] \mathrm{d}t + \left[l_p - p\,\frac{\partial v}{\partial p}\right] \mathrm{d}p \tag{1}$$

to be independent of the path, (2) could be used as an expression of the first law.[1]

[1] This form of the first law was given by Professor R. A. Millikan in a course of thermodynamics at the University of Chicago.

If we choose we may express the energy function in the differential form

$$d\epsilon = \left[c_p - p\frac{\partial v}{\partial t}\right]dt + \left[l_p - p\frac{\partial v}{\partial p}\right]dp$$

where Δt and Δp are the independent variables. This differential expression does not, however, tell the whole story, for the energy function, $\epsilon\ (t, p)$ must also satisfy condition (2) above and is therefore an incomplete expression of § **14.** A statement often used, where condition (2) is satisfied, is that this is a "total (or complete) differential."

17. An analysis of some incomplete statements of the first law that have been used: Physicists have often attempted to express the First Law of Thermodynamics for systems of constant mass in its most general form by writing for it the equations

$$d\varepsilon = dQ + dW$$

or

$$d\varepsilon = dQ - p dv$$

However, as P. W. Bridgman says[1] "The first law is often thought to be the most general of physics, but in a paradoxical sense it is the most special of all laws, because no general meaning can be given to the energy concept, but only specific meaning in special cases. The first law owes its complete generality to the fact that no specific case has yet been found of so broad a character that it cannot be included under one or another special case."

Now let us analyze these equations just set down: What does dW mean? W is not a function of v and p regarded as independent variables but depends on the states through which the system has passed, *i.e.* on the path or curve c (the usage conforming to the definition given earlier). This path or curve c must therefore come first. We have then a one-dimensional range of values and the arc length, s, of this path or curve c affords a natural choice of the independent variable. Then by definition of dv

$$dv = v'(s)\,\Delta s.$$

[1] P. W. Bridgman, The Logic of Modern Physics, p. 127.

Precisely the same remarks apply to **Q**. Like **W**, **Q** depends on the path or curve c chosen and, after this choice has been made, becomes a function of a single variable, as for instance the arc length s.

Now let us suppose for a moment that the first law does not exist. Then we still define ε as the sum of **Q** and **W** along any given curve and we have

$$d\varepsilon = dQ + dW$$

where Δs is the independent variable.

But if the first law does exist we have then the new equation

$$\frac{d\varepsilon}{ds} = \frac{\partial \varepsilon}{\partial p}\frac{dp}{ds} + \frac{\partial \varepsilon}{\partial v}\frac{dv}{ds} = \frac{dQ}{ds} + \frac{dW}{ds}.$$

We still have the equation

$$d\varepsilon = dQ + dW,$$

which is true whether the first law does or does not exist.

If we understand that ε defined as **Q** + **W** is some function of **v** and p then the equation

$$d\varepsilon = dQ + dW$$

involves a physical hypothesis. But to understand that ε is a function of **v** and p presupposes the integral equation for it is the equation that gives **Q** + **W** as independent of the path and hence defining a function of **v** and p which we can name ε. Furthermore if we already have the integral equation we have the first law. Thus in order to get the energy as a function of **v** and p one has to make the physical hypothesis expressed symbolically by the integral equation. In relation to the first law the equation

$$d\varepsilon = dQ + dW$$

is then merely an expression derived from it and since it would be true even though the first law did not exist it can hardly be said to express the first law.

Let us see whether in the equation

$$d\varepsilon = dQ + dW$$

the differentials might be merely approximations for increments, the true equation being

$$\Delta\varepsilon = \Delta Q + \Delta W$$

If this is so, then how are the increments taken? In this view, that the first equation is a near equation for the second equation, the differentials seem self-explanatory: that is, they are close approximations for the increments, and why worry?

How superficial this view is becomes evident on asking—What are the increments? How does the system pass from the state represented by the coordinates (v, p) to the state represented by the coordinates ($v + \Delta v, p + \Delta p$)? For there are an infinite number of possible paths connecting these two states, and some path must be followed, else we have no definite physical picture before us. But as soon as we introduce a path, a curve c through (v, p), we have all the preliminary physical pictures of the preceding text and this is, in fact, the answer. The physical setting described in the explanation of $d\varepsilon = dQ + dW$ is precisely that required for the equation $\Delta\varepsilon = \Delta Q + \Delta W$. When all this is done we can then divide the latter through by Δs, let Δs approach zero and thus deduce $\frac{d\varepsilon}{ds} = \frac{dQ}{ds} + \frac{dW}{ds}$. Multiplying through by ds we get $d\varepsilon = dQ + dW$.

There is no short cut, no self-explanatory method, whereby $d\varepsilon = dQ + dW$ is written down without the intervention of a curve c and the derivatives and differentials pertaining to c.

Physicists also write for the first law

$$\varepsilon = Q + W.$$

If this is all there is to it energy is merely a matter of definition.[1] Where is the physical hypothesis? Some write it

$$\varepsilon_2 - \varepsilon_1 = Q_{12} + W_{12}.$$

[1] On this point Bridgman says (The Logic of Modern Physics, pp. 126-127): "I believe it does not take much examination to convince us that there are no physical operations for measuring dE as such, and that therefore the equation expressing the first law must have a different significance from that which appears on the surface. This is often recognized in the statement that the essence of the first law is that dE is an exact differential determined only by the variables which fix the internal condition of the body, and not a function of the path by which the body is carried from one condition to another."

Either this mode of writing means an abbreviation of our integral equation or else it means nothing except $\boldsymbol{\varepsilon} = \mathbf{Q} + \mathbf{W}$.

18. The second law of thermodynamics. The SECOND LAW of thermodynamics states that

$$\eta\,(t, p) - \eta\,(t_0, p_0) = \int_{t_0,\,p_0}^{t,\,p} \frac{c_p\,dt + l_p\,dp}{\theta},$$

the line integral being extended along any path connecting the points (t_0, p_0) and (t, p), where θ denotes some function of t,

$$\theta = \Gamma\,(t) \neq 0$$

which is the same for all systems, and $\eta\ (t,\ p)$ denotes some function of t and p. This function defined by the preceding equation is called the entropy of the system at the temperature t and pressure p, or of the coordinates (t, p).

We assume, as a physical hypothesis, that for homogeneous systems

$$\lim_{\theta \to 0} \frac{c_p}{\theta} \text{ exists.}$$ [1]

Thus we have entropy as a continuous function of temperature and pressure at and in the neighborhood of $\theta = 0$.

We now extend the definition of entropy, defining it for simple crystalline substances at $\theta = 0$, $p = p_0$ as zero, or

$$\eta\,(t_0, p_0) = 0 \text{ where } \theta = \Gamma\,(t_0) = 0.$$ [2]

19. Differential and partial derivatives of the entropy: Since the line integral is independent of the path we can connect the two limits of integration by any regular curve provided that this curve lies wholly in the simply connected region in which the line integral is defined. Let us integrate over some regular curve from (a, b) to (t, p), then along the straight line from (t, p) to $(t + \Delta t, p)$, p = constant.

[1] This limit is assumed, as a physical hypothesis, to be zero for crystalline substances. Some others believe it to be zero also for liquids.

[2] For justification of this definition see § 93.

$$\eta(t + \Delta t, p) - \eta(t, p) = \text{integral from (a, b) to } (t, p) + \text{integral from } (t, p) \text{ to } (t + \Delta t, p) - \text{integral from (a, b) to } (t, p)$$

$$= \int_{t,\,p}^{t+\Delta t,\,p} \frac{c_p(t, p)\,\mathrm{d}t}{\theta}$$

$$= \Delta t \left[\frac{c_p(t + \alpha\,\Delta t)}{\theta}\right], \quad 0 < \alpha < 1$$

Taking the limit, when Δt approaches zero, of

$$\frac{\eta(t + \Delta t, p) - \eta(t, p)}{\Delta t}$$

gives $\eta_t(t, p) = \frac{c_p}{\theta}$. Similarly $\eta_p(t, p) = \frac{l_p}{\theta}$.

Thus

$$\mathrm{d}\eta = \frac{c_p}{\theta}\,\mathrm{d}t + \frac{l_p}{\theta}\,\mathrm{d}p,$$

where Δt and Δp are the independent variables.

Along any definite path s

$$\frac{\mathrm{d}\eta}{\mathrm{d}s} = \frac{c_p}{\theta}\frac{\mathrm{d}t}{\mathrm{d}s} + \frac{l_p}{\theta}\frac{\mathrm{d}p}{\mathrm{d}s}$$

$$= \frac{1}{\theta}\frac{\mathrm{d}Q}{\mathrm{d}s}$$

Integrating we have

$$Q = \int_{\eta_0} \theta\,\mathrm{d}\eta$$

the line integral being extended along the given path s.

For simple systems of constant mass, homogeneous composition, and having the same temperature and the same pressure at all points and in all directions at a given point we assume as a physical hypothesis that there is a functional relationship between t, p and v which may be expressed as

$$\varphi(t, p, v) = 0.$$

Now

$$\theta = \Gamma(t)$$

We assume as a physical hypothesis that this equation can be solved uniquely for θ

Thus

$$t = \gamma(\theta)$$

Hence

$$\phi\left[p, v, \gamma(\theta)\right] = 0,$$

and

$$\phi(p, v, \theta) = 0,$$

where ϕ represents some function of p, v, θ.

20. Relations between energy derivatives and heat capacities, between entropy derivatives and heat capacities, and between derivatives of heat capacities (all per unit mass): From the first law we have

$$\epsilon_\theta(\theta, p) = c_p - p\frac{\partial v}{\partial \theta} \tag{1}$$

and

$$\epsilon_p(\theta, p) = l_p - p\frac{\partial v}{\partial p}. \tag{2}$$

Thus

$$\left(\frac{\partial c_p}{\partial p}\right)_\theta - p\frac{\partial^2 v}{\partial p\,\partial \theta} - \left(\frac{\partial v}{\partial \theta}\right)_p = \left(\frac{\partial l_p}{\partial \theta}\right)_p - p\frac{\partial^2 v}{\partial \theta\,\partial p} \tag{3}$$

or

$$\left(\frac{\partial c_p}{\partial p}\right)_\theta = \left(\frac{\partial l_p}{\partial \theta}\right)_p + \left(\frac{\partial v}{\partial \theta}\right)_p$$

From the characteristic equation $p = g(v, \theta)$, according to the first law

$$\epsilon(\theta, v) - \epsilon(\theta_0, v_0) = \int_{\theta_0, v_0}^{\theta, v} c_v\, d\theta + l_v\, dv - \int_{\theta_0, v_0}^{\theta, v} p\, dv$$

Hence

$$\epsilon_\theta\,(\theta, v) = c_v \text{ and } \epsilon_v\,(\theta, v) = l_v - p \tag{4, 5}$$

Thus

$$\left(\frac{\partial c_v}{\partial v}\right)_\theta = \left(\frac{\partial l_v}{\partial \theta}\right)_v - \left(\frac{\partial p}{\partial \theta}\right)_v \tag{6}$$

From the second law we have

$$\eta_\theta\,(\theta, p) = \frac{c_p}{\theta} \text{ and } \eta_p\,(\theta, p) = \frac{l_p}{\theta} \tag{7, 8}$$

Thus

$$\left(\frac{\partial \frac{c_p}{\theta}}{\partial p}\right)_\theta = \left(\frac{\partial \frac{l_p}{\theta}}{\partial \theta}\right)_p$$

or

$$\left(\frac{\partial c_p}{\partial p}\right)_\theta = \left(\frac{\partial l_p}{\partial \theta}\right)_p - \frac{l_p}{\theta} \tag{9}$$

According to the second law

$$\eta\,(\theta,v) - \eta\,(\theta_0, v_0) = \int_{\theta_0,\,v_0}^{\theta,\,v} \frac{c_v}{\theta}\,\mathrm{d}\theta + \frac{l_v}{\theta}\,\mathrm{d}v$$

Hence

$$\eta_\theta\,(\theta, v) = \frac{c_v}{\theta} \text{ and } \eta_v\,(\theta, v) = \frac{l_v}{\theta} \tag{10, 11}$$

Thus

$$\left(\frac{\partial \frac{c_v}{\theta}}{\partial v}\right)_\theta = \left(\frac{\partial \frac{l_v}{\theta}}{\partial \theta}\right)_v,$$

or

$$\left(\frac{\partial c_v}{\partial v}\right)_\theta = \left(\frac{\partial l_v}{\partial \theta}\right)_v - \frac{l_v}{\theta} \tag{12}$$

Combining (**20.6**) and (**20.12**) we have

$$l_v = \theta\left(\frac{\partial p}{\partial \theta}\right)_v \tag{13}$$

Substituting the value of l_v in (20.12) we have

$$\left(\frac{\partial c_v}{\partial v}\right)_\theta = \theta\left(\frac{\partial^2 p}{\partial \theta^2}\right)_v \tag{14}$$

Similarly from (20.3) and (20.9)

$$l_p = -\theta\left(\frac{\partial v}{\partial \theta}\right)_p \tag{15}$$

Substituting the value of l_p in (20.9) we have

$$\left(\frac{\partial c_p}{\partial p}\right)_\theta = -\theta\left(\frac{\partial^2 v}{\partial \theta^2}\right)_p \tag{16}$$

DIFFERENCES OF HEAT CAPACITIES PER UNIT MASS

From (20.5)

$$l_v = \left(\frac{\partial \epsilon}{\partial v}\right)_\theta + p$$

From (20.13)

$$l_v = \theta\left(\frac{\partial p}{\partial \theta}\right)_v$$

Hence

$$c_p - c_v = \theta\left(\frac{\partial p}{\partial \theta}\right)_v\left(\frac{\partial v}{\partial \theta}\right)_p \tag{17}$$

Substituting the values of c_v and l_v in (13.8) we have

$$\frac{dQ}{ds} = \left(\frac{\partial \epsilon}{\partial v}\right)_\theta \frac{dv}{ds} + \left(\frac{\partial \epsilon}{\partial \theta}\right)_v \frac{d\theta}{ds} + p\frac{dv}{ds} \tag{18}$$

Substituting the values of c_p and l_p in (13.2) we have

$$\frac{dQ}{ds} = \left(\frac{\partial \epsilon}{\partial \theta}\right)_p \frac{d\theta}{ds} + \left(\frac{\partial \epsilon}{\partial p}\right)_\theta \frac{dp}{ds} + p\left[\left(\frac{\partial v}{\partial \theta}\right)_p \frac{d\theta}{ds} + \left(\frac{\partial v}{\partial p}\right)_\theta \frac{dp}{ds}\right] \tag{19}$$

HOMOGENEOUS ONE-COMPONENT SYSTEMS OF VARIABLE MASS

21. Definitions of component and phase: For the substances or components of which we consider the mass composed we shall choose chemical species or combinations of chemical species.[1]

[1] It would do no good, as we have seen, to choose a dynamic system consisting of electrons and protons as our system for we could not treat it thermodynamically here since our variables would not define the state of such a system.

Furthermore these substances need not have any relation to the internal constitution of the system. They must, however, be so chosen that the masses of each of them in the system, that is, $m_1, \ldots, m_n$, where the number of components is n, are all independent of each other and such that they will express the composition of the homogeneous masses (= phases), over the whole range of states through which we wish the system to pass.

Gibbs calls a substance an *actual component* of a phase when the substance is capable of a continuous increase or decrease in amount in that phase.

He calls a substance a *possible component* of a phase if the substance, though not initially present in this phase, exists in some other phase which is in equilibrium with the first phase and from which the first phase might abstract the substance by a continuous change of concentration. A substance would then be capable only of a continuous increase in amount in the phase in which it exists as a possible component.

The actual components need give us no difficulty, but I am not able to cite any actual examples to illustrate his possible components. In fact some people believe that no such examples of possible components exist. Such a case would occur if we had, say, anhydrous sulfuric acid in equilibrium with some other phase from which it might abstract water. Then water would be a possible component of the phase anhydrous sulfuric acid.

A possible confusion of actual components which might occur is in considering dilute solutions (see Appendix). We have here two situations which depend on our choice of components. For example if we choose $FeCl_3.6H_2O$ (represented on diagram 2 as a vertical dot-dash line), and water as our components, then at a temperature of 310 degrees on the absolute thermodynamic scale and a pressure of one atmosphere a solution composed solely of $FeCl_3.6H_2O$ (i.e. the mass of the water component = 0), will be in equilibrium with crystals of the same composition. Now whether we add water to or subtract water from this liquid phase, keeping temperature and pressure constant, crystals of $FeCl_3.6H_2O$ will go into solution. Here water, for $m_{H_2O} = 0$, is capable of

positive and negative values. This situation will occur when we have a maximum as illustrated in Diag. 2. On the other hand we may so choose our components that the mass of the water component, when zero, is capable only of positive values, i.e. can only increase in amount. Such would be the case if, for example, we chose anhydrous sulfuric acid or chloroform and water as our components, for here water can not decrease below the value zero.

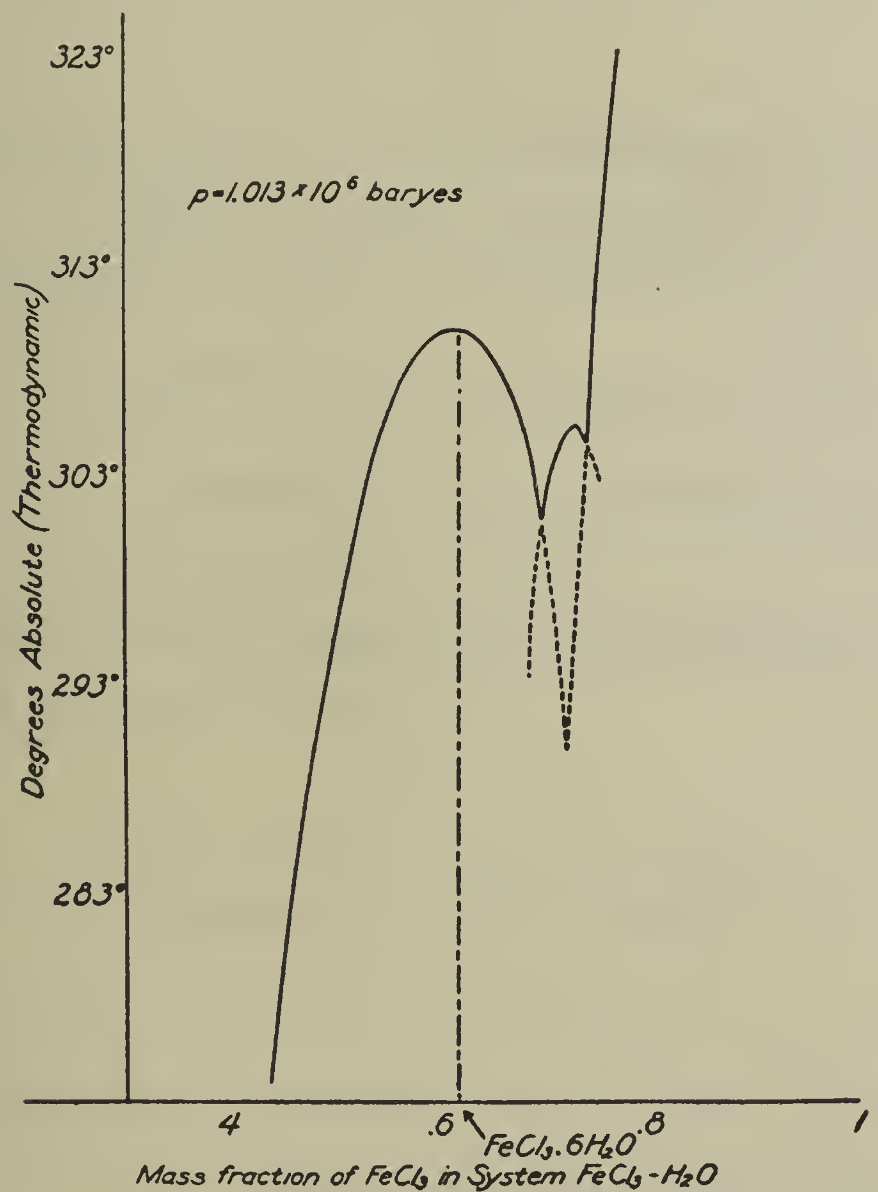

Diagram 2

In neither of these situations does water come under the classification of possible component since we have either added or removed it from the whole system. For water to be a possible component of an anhydrous phase we must have the anhydrous phase in equilibrium at some state with the phase containing water, assuming of course the water can exist as a component of the anhydrous phase. This would be true if the anhydrous phase

were miscible with water over the range of temperature, pressure and concentrations we are considering.

We do not have to consider the internal changes in the system and thus may disregard dissociation in the phase. For example[1] let us consider a potassium chloride-water solution where part of the KCl is dissociated,

$$KCl \rightleftharpoons K^+ + Cl^- \tag{1}$$

Let

μ'' be the chemical potential of the solid potassium chloride,
μ' that of the potassium chloride in solution,
$\mu'_{,}$ that of the undissociated potassium chloride in the solution,
$\mu'_{,,}$ that of the potassium ion in the solution,

and

$\mu'_{,,,}$ that of the chlorine ion in the solution.

Now if we have the potassium chloride-water solution in equilibrium with solid potassium chloride then

$$\mu' = \mu'' \tag{2}$$

but

$$74.56\,\mu'_{,} = 39.1\,\mu'_{,,} + 35.46\,\mu'_{,,,} \tag{3}$$

and thus

$$74.56\,\mu' = 74.56\,(1 - \alpha)\,\mu'_{,} + 39.1\,\alpha\,\mu'_{,,} + 35.46\,\alpha\,\mu'_{,,,} \tag{4}$$

where α denotes the degree of dissociation.

If we multiply (3) by α and subtract (3) and (4) we have

$$\mu' = \mu'_{,}$$

or, from (2),

$$\mu'_{,} = \mu''.$$

[1] This discussion belongs here but since we have not yet defined μ it may be passed over for the time being.

The dissociation constant has thus dropped out of the equations. Furthermore we can not obtain the amount of dissociation from this kind of reasoning. Thus in order to determine the potentials of the ions one of the things we must have given us is the degree of dissociation.

Each homogeneous part of a mass we shall call a phase and thus the number of phases will be determined by the number of parts of the mass considered that differ in composition or state or both. Crystalline $FeCl_3.6H_2O$ in equilibrium with crystalline $FeCl_3.3\frac{1}{2}H_2O$ is an illustration of two phases that differ in composition; ice in equilibrium with water is an illustration of two phases that differ in state; a water solution of sodium chloride in equilibrium with water vapor is an illustration of two phases that differ both in composition and state.

Thus a phase is a geometrically connected subdivision of a physical system in which each one of the components is physically distinguishable from the same component in the other phases.

22. Definitions of heat and work: The work, **W**, done on the system we define by the equation

$$\mathbf{W} = -\int_{t_0,\, p_0,\, \mathrm{m}_0}^{t,\, p,\, \mathrm{m}} p\, \mathrm{dv} = -\int_{t_0,\, p_0,\, \mathrm{m}_0}^{t,\, p,\, \mathrm{m}} \mathbf{m}\, p \frac{\partial v}{\partial t} \mathrm{d}t + \mathbf{m}\, p \frac{\partial v}{\partial p} \mathrm{d}p + p\, v\, \mathrm{d}\mathbf{m}$$

and the heat, **Q**, received by the equation

$$\mathbf{Q} = \int_{t_0,\, p_0,\, \mathrm{m}_0}^{t,\, p,\, \mathrm{m}} \mathbf{m}\, c_p\, \mathrm{d}t + \mathbf{m}\, l_p\, \mathrm{d}p + f\, \mathrm{d}\mathbf{m}$$

where c_p, l_p and f denote functions of t and p. This is really an extension of the ordinary definition of heat. To a physicist f here has meaning in terms of heat received *measured as such* only when there is an interchange of heat between different parts of the system; for example if we had two phases present, one increasing and the other decreasing. The physical significance of f is indicated in § 24.

23. The first law of thermodynamics: Then the first law of thermodynamics is expressed by the equation

$$\boldsymbol{\varepsilon}(t, p, \mathrm{m}) - \boldsymbol{\varepsilon}(t_0, p_0, \mathrm{m}_0) = \int_{t_0,\, p_0,\, \mathrm{m}_0}^{t,\, p,\, \mathrm{m}} \mathrm{m}\left(c_p - p\frac{\partial v}{\partial t}\right)\mathrm{d}t$$

$$+ \mathrm{m}\left(l_p - p\frac{\partial v}{\partial p}\right)\mathrm{d}p + (\mu + f - pv)\,\mathrm{dm} \qquad (1)$$

where μ denotes a function of the temperature and pressure.

In the special case where the mass, m, remains constant we have

$$\boldsymbol{\varepsilon}(t, p, \mathrm{m}) - \boldsymbol{\varepsilon}(t_0, p_0, \mathrm{m}) = \mathrm{m}\int_{t_0,\, p_0}^{t,\, p} \left(c_p - p\frac{\partial v}{\partial t}\right)\mathrm{d}t$$

$$+ \left(l_p - p\frac{\partial v}{\partial p}\right)\mathrm{d}p. \qquad (2)$$

Now by definition

$$\boldsymbol{\varepsilon} = \mathrm{m}\,\epsilon,$$

then

$$\boldsymbol{\varepsilon}(t_0, p_0, \mathrm{m}) = \mathrm{m}\,\epsilon(t_0, p_0) = 0.$$

Thus

$$\boldsymbol{\varepsilon}(t_0, p_0, 0) = 0,$$

and hence

$$\boldsymbol{\varepsilon}(t, p, 0) = 0. \qquad (3)$$

Now

$$\boldsymbol{\varepsilon}(t, p, \mathrm{m}) - \boldsymbol{\varepsilon}(t, p, 0) = \int_{t,\, p,\, 0}^{t,\, p,\, \mathrm{m}} (\mu + f - pv)\,\mathrm{dm}, \qquad (4)$$

or

$$\boldsymbol{\varepsilon}(t, p, \mathrm{m}) = (\mu + f - pv)\,\mathrm{m}, \qquad (5)$$

and by definition

$$\epsilon = \frac{\boldsymbol{\varepsilon}}{\mathrm{m}},$$

then

$$\epsilon = \mu + f - pv \qquad (6)$$

24. The second law of thermodynamics: The second law is expressed by the equation

$$\mathbf{n}(t, p, \mathrm{m}) - \mathbf{n}(t_0, p_0, \mathrm{m}_0) = \int_{t_0, p_0, \mathrm{m}_0}^{t, p, \mathrm{m}} \mathrm{m} \frac{c_p}{\theta} \mathrm{d}t + \mathrm{m} \frac{l_p}{\theta} \mathrm{d}p + \frac{f}{\theta} \mathrm{dm} \quad (1)$$

where θ denotes some function of t, $\theta = \Gamma(t)$, *which is the same for all systems.*

In the special case where the mass, m, remains constant we have

$$\mathbf{n}(t, p, \mathrm{m}) - \mathbf{n}(t_0, p_0, \mathrm{m}) = \mathrm{m} \int_{t_0, p_0}^{t, p} \frac{c_p}{\theta} \mathrm{d}t + \frac{l_p}{\theta} \mathrm{d}p \quad (2)$$

$$= \mathrm{m}\, \eta(t, p) - \mathrm{m}\, \eta(t_0, p_0).$$

Now we have defined the entropy of a unit mass simple crystalline substance as zero at $\theta = \Gamma(t_0) = 0$ and $p = p_0$,

$$\eta(t_0, p_0) = 0$$

and since, by definition,

$$\mathbf{n} = \mathrm{m}\, \eta$$

we have

$$\mathbf{n}(t_0, p_0, \mathrm{m}) = \mathrm{m}\, \eta(t_0, p_0) = 0$$

Thus

$$\mathbf{n}(t_0, p_0, 0) = 0$$

and hence

$$\mathbf{n}(t, p, 0) = 0. \quad (3)$$

Now

$$\mathbf{n}(t, p, \mathrm{m}) - \mathbf{n}(t, p, 0) = \int_{t, p, 0}^{t, p, \mathrm{m}} \frac{f}{\theta} \mathrm{dm}$$

or

$$\mathbf{n}(t, p, \mathrm{m}) = \frac{f(t, p)}{\theta} \mathrm{m}, \quad (4)$$

and by definition

$$\eta = \frac{\mathbf{n}}{\mathbf{m}},$$

then

$$\eta = \frac{f(t, p)}{\theta} \tag{5}$$

But we have $\epsilon = \mu + f - pv$, from (**23**.6),

then from the relation

$$\eta = \frac{f}{\theta},$$

we obtain

$$\epsilon = \mu + \theta\,\eta - pv$$

or

$$\mu = \epsilon - \theta\,\eta + pv. \tag{6}$$

25. Definitions of Gibbs' thermodynamic functions: Gibbs has defined three additional thermodynamic functions $\boldsymbol{\zeta}$, $\boldsymbol{\chi}$ and $\boldsymbol{\psi}$ by the equations

$$\boldsymbol{\zeta} = \boldsymbol{\varepsilon} + p\mathbf{v} - \theta\,\mathbf{n}$$

$$\boldsymbol{\chi} = \boldsymbol{\varepsilon} + p\mathbf{v}$$

$$\boldsymbol{\psi} = \boldsymbol{\varepsilon} - \theta\,\mathbf{n}$$

We denote the values of the functions for unit mass by ζ, χ and ψ, that is, by definition

$$\zeta = \frac{\boldsymbol{\zeta}}{\mathbf{m}}$$

$$\chi = \frac{\boldsymbol{\chi}}{\mathbf{m}}, \text{ and}$$

$$\psi = \frac{\boldsymbol{\psi}}{\mathbf{m}}.$$

Therefore we have from (**24**.6) for the one-component single phase system

$$\mu = \zeta.$$

CHAPTER III

Homogeneous binary systems of variable mass and composition

Let us now consider systems variable in mass and composition in successive states but homogeneous in each state, that is, in each state the composition, density and temperature are the same at all points, the pressure is the same at all points and is the same in all directions at any given point. Let us consider first a homogeneous system composed of two components.

26. Definitions of specific volume and mass fraction: The specific volume, v, is defined by the equation

$$v = \frac{\mathbf{v}}{\mathbf{m}_1 + \mathbf{m}_2}, \qquad \begin{array}{c} 0 < \mathbf{v} < \infty \\ 0 \leqq \mathbf{m}_1 < \infty \\ 0 \leqq \mathbf{m}_2 < \infty \\ \mathbf{m}_1 + \mathbf{m}_2 \neq 0. \end{array}$$

where $\mathbf{v}$ denotes the total volume of the system, $\mathbf{m}_1$ the mass of substance s_1, and $\mathbf{m}_2$ the mass of substance s_2. We define the mass fraction of the first component, m_1, by the equation

$$m_1 = \frac{\mathbf{m}_1}{\mathbf{m}_1 + \mathbf{m}_2}, \qquad \begin{array}{c} 0 \leqq \mathbf{m}_1 < \infty \\ 0 \leqq \mathbf{m}_2 < \infty \\ \mathbf{m}_1 + \mathbf{m}_2 \neq 0. \end{array}$$

and similarly the mass fraction m_2 of the second component by

$$m_2 = \frac{\mathbf{m}_2}{\mathbf{m}_1 + \mathbf{m}_2}.$$

Hence

$$m_1 + m_2 = 1.$$

27. Equation of state: We assume as a physical hypothesis that when any three of the properties, pressure, specific volume,

temperature, fraction of substance s_1, of a particular system are given the fourth is uniquely determined, or that

$$\varphi(p, v, t, m_1) = 0.$$

Thus

$$\mathbf{v} = (\mathbf{m}_1 + \mathbf{m}_2)\, v\,(t, p, m_1).$$

We assume further

$$\mathbf{v} = 0,\ t = t_0,\ p = p_0,\ \mathbf{m}_1 = 0,\ \mathbf{m}_2 = 0.$$

28. Definitions of work and heat: The work, W, done on the system is defined by the equation

$$\mathbf{W} = -\int_{t_0,\, p_0,\, \mathbf{m}_{1_0},\, \mathbf{m}_{2_0}}^{t,\, p,\, \mathbf{m}_1,\, \mathbf{m}_2} p\,d\mathbf{v} \tag{1}$$

where

$$\mathbf{v} = (\mathbf{m}_1 + \mathbf{m}_2)\, v\,(t, p, m_1).$$

Expanding, we have

$$\mathbf{W} = -\int_{t_0,\, p_0,\, \mathbf{m}_{1_0},\, \mathbf{m}_{2_0}}^{t,\, p,\, \mathbf{m}_1,\, \mathbf{m}_2} (\mathbf{m}_1 + \mathbf{m}_2)\, p \frac{\partial v}{\partial t} dt + (\mathbf{m}_1 + \mathbf{m}_2)\, p \frac{\partial v}{\partial p} dp$$

$$+ p\left(v + m_2 \frac{\partial v}{\partial m_1}\right) d\mathbf{m}_1 + p\left(v - m_1 \frac{\partial v}{\partial m_1}\right) d\mathbf{m}_2. \tag{2}$$

In the special case where $\mathbf{m}_1 + \mathbf{m}_2 = 1$, i.e. in the unit mass case, this integral reduces to

$$W = -\int_{t_0,\, p_0,\, m_{1_0}}^{t,\, p,\, m_1} p \frac{\partial v}{\partial t} dt + p \frac{\partial v}{\partial p} dp + p \frac{\partial v}{\partial m_1} dm_1. \tag{3}$$

The heat, Q, received by the system is defined by the equation

$$\mathbf{Q}^{(1)} = \int_{t_0,\, p_0,\, \mathbf{m}_{1_0},\, \mathbf{m}_{2_0}}^{t,\, p,\, \mathbf{m}_1,\, \mathbf{m}_2} (\mathbf{m}_1 + \mathbf{m}_2)\, c_p\, dt + (\mathbf{m}_1 + \mathbf{m}_2)\, l_p\, dp + l_{m_1} d\mathbf{m}_1 + l_{m_2} d\mathbf{m}_2 \tag{4}$$

where c_p, l_p, l_{m_1} and l_{m_2} are continuous functions of t, p, and m_1.

[1] If t is expressed in centigrade degrees on the scale of the hydrogen gas thermometer, p in dynes per cm^2, v in cubic centimeters, $\mathbf{m}_1$ and $\mathbf{m}_2$ in grams, then Q and W will be expressed in ergs.

In the special case where $m_1 + m_2 = 1$, i.e. in the unit mass case, this integral reduces to

$$Q = \int_{t_0,\, p_0,\, m_{1_0}}^{t,\, p,\, m_1} c_p\, dt + l_p dp + (l_{m_1} - l_{m_2})\, dm_1 \tag{5}$$

29. Definitions of $c_p, l_p, l_{m_1} - l_{m_2}$: Along the path $s = t, p = K'$, $m_1 = K''$
where K' and K'' are constants

$\frac{dQ}{ds} = \frac{dQ}{dt}$ which is defined as the heat capacity per unit mass at constant pressure and concentration.
Thus

$$\frac{dQ}{dt} = c_p, \tag{1}$$

and along this path

$$\frac{dW}{ds} = \frac{dW}{dt} = -p\frac{\partial v}{\partial t}. \tag{2}$$

Along the path $t = K, s = p, m_1 = K''$
where K and K'' are constants

$\frac{dQ}{ds} = \frac{dQ}{dp}$ which is defined as the latent heat of change of pressure per unit mass at constant temperature and concentration.
Thus

$$\frac{dQ}{dp} = l_p \tag{3}$$

and along this path

$$\frac{dW}{ds} = \frac{dW}{dp} = -p\frac{\partial v}{\partial p}. \tag{4}$$

Along the path $t = K,\ p = K',\ s = m_1$

$\frac{dQ}{ds} = \frac{dQ}{dm_1}$ which is defined as the heat of change of concentration at constant pressure and temperature.

Thus

$$\frac{dQ}{dm_1} = l_{m_1} - l_{m_2} \tag{5}$$

and along this path

$$\frac{dW}{ds} = \frac{dW}{dm_1} = -p\frac{\partial v}{\partial m_1}. \tag{6}$$

30. Transformation of the heat and work integrals: Now by hypothesis p can be expressed as a function of t, v, and m_1

$$p = f(t, v, m_1) \tag{1}$$

Thus

$$\int_{t_0,\, p_0,\, m_{1_0}}^{t,\, p,\, m_1} c_p\, dt + l_p\, dp + (l_{m_1} - l_{m_2})\, dm_1 = \int_{t_0,\, v_0,\, m_{1_0}}^{t,\, v,\, m_1} \left(c_p + l_p\frac{\partial p}{\partial t}\right) dt + l_p\frac{\partial p}{\partial v} dv + \left(l_{m_1} - l_{m_2} + l_p\frac{\partial p}{\partial m_1}\right) dm_1 \tag{2}$$

Let us define

$$c_p + l_p\frac{\partial p}{\partial t} = c_v \tag{3}$$

$$l_p\frac{\partial p}{\partial v} = l_v \tag{4}$$

$$l_{m_1} - l_{m_2} + l_p\frac{\partial p}{\partial m_1} = l_{W_1} - l_{W_2} \tag{5}$$

then

$$\int_{t_0,\, p_0,\, m_{1_0}}^{t,\, p,\, m_1} c_p\, dt + l_p\, dp + (l_{m_1} - l_{m_2})\, dm_1 = \int_{t_0,\, v_0,\, m_{1_0}}^{t,\, v,\, m_1} c_v\, dt + l_v\, dv + (l_{W_1} - l_{W_2})\, dm_1 \tag{6}$$

The corresponding transformation of the work integral yields nothing new, as we have seen,

$$-\int_{t_0,\, p_0,\, m_{1_0}}^{t,\, p,\, m_1} p\frac{\partial v}{\partial t} dt + p\frac{\partial v}{\partial p} dp + p\frac{\partial v}{\partial m_1} dm_1 = -\int_{t_0,\, v_0,\, m_{1_0}}^{t,\, v,\, m_1} p\, dv \tag{7}$$

31. Definitions of $c_v, l_v, l_{W_1} - l_{W_2}$: Along the path $t = \mathrm{s}, v = \mathrm{K}'''$, $m_1 = \mathrm{K}''$
where K''' and K'' are constants

$\frac{dQ}{ds} = \frac{dQ}{dt}$ which is defined as the heat capacity per unit mass at constant volume and concentration.

Thus

$$\frac{dQ}{dt} = c_v \tag{1}$$

and along this path

$$\frac{dW}{ds} = \frac{dW}{dt} = 0. \tag{2}$$

Along the path $t = \mathrm{K}$, $v = \mathrm{s}$, $m_1 = \mathrm{K}''$
where K and K'' are constants

$\frac{dQ}{ds} = \frac{dQ}{dv}$ which is defined as the latent heat of change of volume per unit mass at constant temperature and concentration.

Thus

$$\frac{dQ}{dv} = l_v \tag{3}$$

and along this path

$$\frac{dW}{ds} = \frac{dW}{dv} = -p. \tag{4}$$

Along the path $t = \mathrm{K}$, $v = \mathrm{K}'''$, $m_1 = \mathrm{s}$

$\frac{dQ}{ds} = \frac{dQ}{dm_1}$ which is defined as the heat of change of concentration at constant temperature and volume.

Thus

$$\frac{dQ}{dm_1} = l_{W_1} - l_{W_2} \tag{5}$$

and along this path

$$\frac{dW}{ds} = \frac{dW}{dm_1} = 0. \tag{6}$$

32. The first law of thermodynamics: The first law of thermodynamics is expressed by the equation

$$\boldsymbol{\varepsilon}(t, p, \mathbf{m}_1, \mathbf{m}_2) - \boldsymbol{\varepsilon}(t_0, p_0, \mathbf{m}_{1_0}, \mathbf{m}_{2_0}) =$$

$$\int_{t_0,\, p_0,\, \mathbf{m}_{1_0},\, \mathbf{m}_{2_0}}^{t,\, p,\, \mathbf{m}_1,\, \mathbf{m}_2} \left(\mathbf{m}_1 + \mathbf{m}_2\right)\left(c_p - p\frac{\partial v}{\partial t}\right) \mathrm{d}t + \left(\mathbf{m}_1 + \mathbf{m}_2\right)\left(l_p - p\frac{\partial v}{\partial p}\right) \mathrm{d}p +$$

$$\left[l_{m_1} - pm_2\frac{\partial v}{\partial m_1} - pv + \mu_1\right] \mathrm{d}\mathbf{m}_1 +$$

$$\left[l_{m_2} + pm_1\frac{\partial v}{\partial m_1} - pv + \mu_2\right] \mathrm{d}\mathbf{m}_2$$

where μ_1 and μ_2 denote continuous functions of t, p, and m_1.

SPECIAL CASES

(1) In the special case where $\mathbf{m}_1$ and $\mathbf{m}_2$ are constant the first law reduces to the following equation:

$$\boldsymbol{\varepsilon}(t, p, \mathbf{m}_1, \mathbf{m}_2) - \boldsymbol{\varepsilon}(t_0, p_0, \mathbf{m}_1, \mathbf{m}_2) = (\mathbf{m}_1 + \mathbf{m}_2) \int_{t_0,\, p_0,\, \mathbf{m}_1,\, \mathbf{m}_2}^{t,\, p,\, \mathbf{m}_1,\, \mathbf{m}_2} \left(c_p - p\frac{\partial v}{\partial t}\right) \mathrm{d}t$$

$$+ \left(l_p - p\frac{\partial v}{\partial p}\right) \mathrm{d}p. \qquad (2)$$

Now $\boldsymbol{\varepsilon}(t, p, \mathbf{m}_1, 0) = \boldsymbol{\varepsilon}'(t, p, \mathbf{m})$
and $\boldsymbol{\varepsilon}'(t, p, 0) = 0$ from (23.3)
Thus

$$\boldsymbol{\varepsilon}(t, p, 0, 0) = 0 \qquad (4)$$

(2) In the special case where t, p, m_1 are constant the first law reduces to the following equation:

$$\boldsymbol{\varepsilon}(t, p, \mathbf{m}_1, \mathbf{m}_2) - \boldsymbol{\varepsilon}(t, p, 0, 0) = \left(l_{m_1} - pm_2\frac{\partial v}{\partial m_1} - pv + \mu_1\right)\mathbf{m}_1$$

$$+ \left(l_{m_2} + pm_1\frac{\partial v}{\partial m_1} - pv + \mu_2\right)\mathbf{m}_2$$

$$= \mathbf{m}_1 l_{m_1} + \mathbf{m}_2 l_{m_2} - pv(\mathbf{m}_1 + \mathbf{m}_2)$$

$$+ \mu_1 \mathbf{m}_1 + \mu_2 \mathbf{m}_2 \qquad (5)$$

Let $\epsilon = \dfrac{\boldsymbol{\varepsilon}}{\mathbf{m}_1 + \mathbf{m}_2}$ by definition,

Then

$$\begin{aligned}\epsilon &= m_1\, l_{m_1} + m_2\, l_{m_2} - pv + \mu_1\, m_1 + \mu_2\, m_2 \\ &= \epsilon\,(t, p, m_1)\end{aligned} \tag{6}$$

Hence

$$\boldsymbol{\varepsilon}\,(t, p, \mathbf{m}_1, \mathbf{m}_2) = (\mathbf{m}_1 + \mathbf{m}_2)\,\epsilon\,(t, p, m_1). \tag{7}$$

(3) In the special case where $\mathbf{m}_1 + \mathbf{m}_2 = 1$, i.e. the unit mass case, the first law reduces to the following equation:

$$\begin{aligned}\epsilon\,(t, p, m_1) - \epsilon\,(t_0, p_0, m_{1_0}) &= \int_{t_0,\, p_0,\, m_{1_0}}^{t,\, p,\, m_1} \left[c_p - p\frac{\partial v}{\partial t}\right]\mathrm{d}t + \left[l_p - p\frac{\partial v}{\partial p}\right]\mathrm{d}p \\ &\quad + \left[\left(l_{m_1} - pm_2\frac{\partial v}{\partial m_1} - pv + \mu_1\right)\right. \\ &\quad \left. - \left(l_{m_2} + pm_1\frac{\partial v}{\partial m_1} - pv + \mu_2\right)\right]\mathrm{d}m_1 \\ &= \int_{t_0,\, p_0,\, m_{1_0}}^{t,\, p,\, m_1} \left[c_p - p\frac{\partial v}{\partial t}\right]\mathrm{d}t + \left[l_p - p\frac{\partial v}{\partial p}\right]\mathrm{d}p \\ &\quad + \left[l_{m_1} - l_{m_2} - p\frac{\partial v}{\partial m_1} + \mu_1 - \mu_2\right]\mathrm{d}m_1\end{aligned} \tag{8}$$

Hence

$$\epsilon_t\,(t, p, m_1) = c_p - p\left(\frac{\partial v}{\partial t}\right)_{p,\, m_1} \tag{9}$$

$$\epsilon_p\,(t, p, m_1) = l_p - p\left(\frac{\partial v}{\partial p}\right)_{t,\, m_1} \tag{10}$$

$$\epsilon_{m_1}\,(t, p, m_1) = l_{m_1} - l_{m_2} - p\frac{\partial v}{\partial m_1} + \mu_1 - \mu_2 \tag{11}$$

Now letting $m_1 = 0$, then $m_2 = 1$

and $\epsilon\,(t_0, p_0, 0) = \epsilon'\,(t_0, p_0) = 0$; similarly if $m_1 = 1$, then $m_2 = 0$ and $\epsilon\,(t_0, p_0, 1) = \epsilon''\,(t_0, p_0) = 0$ (12)

since we have already defined

$$\epsilon' (t_0, p_0) = \epsilon'' (t_0, p_0) = 0$$

Further,

$$\frac{\partial \epsilon}{\partial t \, \partial p} = \frac{\partial \epsilon}{\partial p \, \partial t}$$

or

$$\left(\frac{\partial l_p}{\partial t}\right)_{p,\, m_1} - p \frac{\partial^2 v}{\partial t \, \partial p} = \left(\frac{\partial c_p}{\partial p}\right)_{t,\, m_1} - \frac{\partial v}{\partial t} - p \frac{\partial^2 v}{\partial p \, \partial t}$$

thus

$$\left(\frac{\partial c_p}{\partial p}\right)_{t,\, m_1} = \left(\frac{\partial l_p}{\partial t}\right)_{p,\, m_1} + \left(\frac{\partial v}{\partial t}\right)_{p,\, m_1}. \tag{13}$$

Similarly

$$\frac{\partial (l_{m_1} - l_{m_2})}{\partial t} - p \frac{\partial v}{\partial t \, \partial m_1} + \frac{\partial (\mu_1 - \mu_2)}{\partial t} = \frac{\partial c_p}{\partial m_1} - p \frac{\partial v}{\partial m_1 \, \partial t}$$

thus

$$\left(\frac{\partial c_p}{\partial m_1}\right)_{t,\, p} = \frac{\partial l_{m_1}}{\partial t} - \frac{\partial l_{m_2}}{\partial t} + \frac{\partial \mu_1}{\partial t} - \frac{\partial \mu_2}{\partial t}; \tag{14}$$

and

$$\frac{\partial l_{m_1}}{\partial p} - \frac{\partial l_{m_2}}{\partial p} - \frac{\partial v}{\partial m_1} - p \frac{d^2 v}{\partial p \, \partial m_1} + \frac{\partial \mu_1}{\partial p} - \frac{\partial \mu_2}{\partial p} = \frac{\partial l_p}{\partial m_1} - p \frac{\partial^2 v}{\partial m_1 \, \partial p}$$

thus

$$\frac{\partial l_p}{\partial m_1} = \frac{\partial l_{m_1}}{\partial p} - \frac{\partial l_{m_2}}{\partial p} - \frac{dv}{\partial m_1} + \frac{\partial \mu_1}{\partial p} - \frac{\partial \mu_2}{\partial p} \tag{15}$$

33. Transformation of the energy integral. By hypothesis

$$p = p\,(t, v, m_1)$$

Thus

$$\int_{t_0,\, p_0,\, m_{1_0}}^{t,\, p,\, m_1} \left[c_p - p \frac{\partial v}{\partial t}\right] dt + \left[l_p - p \frac{\partial v}{\partial p}\right] dp$$
$$+ \left[l_{m_1} - l_{m_2} - p \frac{\partial v}{\partial m_1} + \mu_1 - \mu_2\right] dm_1$$

$$= \int_{t_0, v_0, m_{1_0}}^{t, v, m_1} \left[c_p - p \frac{\partial v}{\partial t} + l_p \frac{\partial p}{\partial t} - p \frac{\partial v}{\partial p} \frac{\partial p}{\partial t} \right] \mathrm{d}t + \left[l_p \frac{\partial p}{\partial v} - p \frac{\partial v}{\partial p} \frac{\partial p}{\partial v} \right] \mathrm{d}v$$

$$+ \left[l_{m_1} - l_{m_2} - p \frac{\partial v}{\partial m_1} + \mu_1 - \mu_2 + l_p \frac{\partial p}{\partial m_1} - p \frac{\partial v}{\partial p} \frac{\partial p}{\partial m_1} \right] \mathrm{d}m_1$$

$$= \int_{t_0, v_0, m_{1_0}}^{t, v, m_1} c_v \, \mathrm{d}t + (l_v - p) \, \mathrm{d}v + (l_{W_1} - l_{W_2} + \mu_1 - \mu_2) \, \mathrm{d}m_1 \tag{1}$$

since, by definitión, § 30, we have

$$c_p + l_p \frac{\partial p}{\partial t} = c_v$$

$$l_p \frac{\partial p}{\partial v} = l_v$$

$$l_{m_1} - l_{m_2} + l_p \frac{\partial p}{\partial m_1} = l_{W_1} - l_{W_2}$$

Hence

$$\frac{\partial \epsilon}{\partial t} = c_v$$

$$\frac{\partial \epsilon}{\partial v} = l_v - p$$

$$\frac{\partial \epsilon}{\partial m_1} = l_{W_1} - l_{W_2} + \mu_1 - \mu_2$$

Thus

$$\frac{\partial c_v}{\partial v} = \frac{\partial l_v}{\partial t} - \frac{\partial p}{\partial t}$$

$$\frac{\partial c_v}{\partial m_1} = \frac{\partial l_{W_1}}{\partial t} - \frac{\partial l_{W_2}}{\partial t} + \frac{\partial \mu_1}{\partial t} - \frac{\partial \mu_2}{\partial t}$$

$$\frac{\partial l_v}{\partial m_1} - \frac{\partial p}{\partial m_1} = \frac{\partial l_{W_1}}{\partial v} - \frac{\partial l_{W_2}}{\partial v} + \frac{\partial \mu_1}{\partial v} - \frac{\partial \mu_2}{\partial v}$$

34. The second law of thermodynamics: The second law is expressed by the equation

$$\mathbf{n}(t, p, \mathrm{m}_1, \mathrm{m}_2) - \mathbf{n}(t_0, p_0, \mathrm{m}_{1_0}, \mathrm{m}_{2_0}) = \int_{t_0,\, p_0,\, \mathrm{m}_{1_0},\, \mathrm{m}_{2_0}}^{t,\, p,\, \mathrm{m}_1,\, \mathrm{m}_2} (\mathrm{m}_1 + \mathrm{m}_2) \frac{c_p}{\theta} \mathrm{d}t$$

$$+ (\mathrm{m}_1 + \mathrm{m}_2) \frac{l_p}{\theta} \mathrm{d}p + \frac{l_{m_1}}{\theta} \mathrm{dm}_1 + \frac{l_{m_2}}{\theta} \mathrm{dm}_2 \tag{2}$$

where θ is a function of t only, $\theta = \Gamma(t)$, the same for all systems.

SPECIAL CASES

(1) In the special case where m_1 and m_2 are constant the equation reduces to the following

$$\mathbf{n}(t, p, \mathrm{m}_1, \mathrm{m}_2) - \mathbf{n}(t_0, p_0, \mathrm{m}_1, \mathrm{m}_2) = (\mathrm{m}_1 + \mathrm{m}_2) \int_{t_0,\, p_0,\, \mathrm{m}_1,\, \mathrm{m}_2}^{t,\, p,\, \mathrm{m}_1,\, \mathrm{m}_2} \frac{c_p}{\theta} \mathrm{d}t + \frac{l_p}{\theta} \mathrm{d}p \tag{3}$$

Now $\mathbf{n}(t, p, \mathrm{m}_1, 0) = \mathbf{n}'(t, p, \mathrm{m})$
and $\mathbf{n}'(t, p, 0) = 0$ by (24.3).
Then

$$\mathbf{n}(t, p, 0, 0) = 0 \tag{4}$$

(2) In the special case where t, p and m_1 are constant

$$\mathbf{n}(t, p, \mathrm{m}_1, \mathrm{m}_2) - \mathbf{n}(t, p, 0, 0) = \mathrm{m}_1 \frac{l_{m_1}}{\theta} + \mathrm{m}_2 \frac{l_{m_2}}{\theta} \tag{5}$$

Hence

$$\mathbf{n}(t, p, \mathrm{m}_1, \mathrm{m}_2) = \mathrm{m}_1 \frac{l_{m_1}}{\theta} + \mathrm{m}_2 \frac{l_{m_2}}{\theta} \tag{6}$$

Let $\eta = \dfrac{\mathbf{n}}{\mathrm{m}_1 + \mathrm{m}_2}$ by definition

Then

$$\eta = m_1 \frac{l_{m_1}}{\theta} + m_2 \frac{l_{m_2}}{\theta} \tag{7}$$

$$= \eta(t, p, m_1)$$

Thus

$$\mathfrak{n}(t, p, \mathbf{m}_1, \mathbf{m}_2) = (\mathbf{m}_1 + \mathbf{m}_2)\,\eta\,(t, p, m_1) \tag{8}$$

Further we have

$$\begin{aligned} \epsilon &= m_1\, l_{m_1} + m_2\, l_{m_2} - pv + \mu_1 m_1 + \mu_2 m_2 \\ &= \theta\,\eta - pv + \mu_1 m_1 + \mu_2 m_2 \text{ [1]} \end{aligned} \tag{9}$$

(3) In the special case where $\mathbf{m}_1 + \mathbf{m}_2 = 1$, i.e. in the unit mass case, the second law of thermodynamics for binary systems reduces to

$$\eta\,(t, p, m_1) - \eta\,(t_0, p_0, m_{1_0}) = \int_{t_0,\, p_0,\, m_{1_0}}^{t,\, p,\, m_1} \frac{c_p}{\theta}\,dt + \frac{l_p}{\theta}\,dp + \frac{l_{m_1} - l_{m_2}}{\theta}\,dm_1 \tag{10}$$

Further $\eta\,(t_0, p_0, 0) = \eta'\,(t_0, p_0) = 0$ and $\eta\,(t_0, p_0, 1) = \eta''\,(t_0, p_0) = 0$ since we have already defined $\eta'\,(t_0, p_0) = \eta''\,(t_0, p_0) = 0$.
Hence

$$\frac{\partial \eta}{\partial t} = \frac{c_p}{\theta} \tag{11}$$

$$\frac{\partial \eta}{\partial p} = \frac{l_p}{\theta} \tag{12}$$

$$\frac{\partial \eta}{\partial m_1} = \frac{l_{m_1}}{\theta} - \frac{l_{m_2}}{\theta} \tag{13}$$

Thus

$$\frac{1}{\theta}\frac{\partial c_p}{\partial p} = \frac{1}{\theta}\frac{\partial l_p}{\partial t} - \frac{l_p}{\theta^2},$$

and therefore

$$\left(\frac{\partial v}{\partial t}\right)_{p,\, m_1} = -\frac{l_p}{\theta}; \tag{14}$$

$$\frac{1}{\theta}\frac{\partial c_p}{\partial m_1} = \frac{1}{\theta}\frac{\partial\,(l_{m_1} - l_{m_2})}{\partial t} - \frac{(l_{m_1} - l_{m_2})}{\theta^2}$$

[1] This is Gibbs' equation 93 for a binary system of unit mass.

and therefore

$$\frac{\partial \mu_1}{\partial t} - \frac{\partial \mu_2}{\partial t} = \frac{l_{m_1}}{\theta} - \frac{l_{m_2}}{\theta}; \tag{15}$$

$$\frac{1}{\theta}\frac{\partial l_p}{\partial m_1} = \frac{1}{\theta}\frac{\partial\,(l_{m_1} - l_{m_2})}{\partial p}$$

and therefore

$$\frac{\partial v}{\partial m_1} = \frac{\partial \mu_1}{\partial p} - \frac{\partial \mu_2}{\partial p} \tag{16}$$

35. Transformation of the entropy integral: By hypothesis

$$p = p\,(t, v, m_1),$$

Thus

$$\int_{t_0,\, p_0,\, m_{1_0}}^{t,\, p,\, m_1} \frac{c_p}{\theta}\,\mathrm{d}t + \frac{l_p}{\theta}\,\mathrm{d}p + \frac{l_{m_1} - l_{m_2}}{\theta}\,\mathrm{d}m_1$$

$$= \int_{t_0,\, v_0,\, m_{1_0}}^{t,\, v,\, m_1} \left[\frac{c_p}{\theta} + \frac{l_p}{\theta}\frac{\partial p}{\partial t}\right]\mathrm{d}t + \frac{l_p}{\theta}\frac{\partial p}{\partial v}\,\mathrm{d}v + \left[\frac{l_{m_1} - l_{m_2}}{\theta} + \frac{l_p}{\theta}\frac{\partial p}{\partial m_1}\right]\mathrm{d}m_1 \tag{1}$$

and since, by definition, we have

$$c_p + l_p \frac{\partial p}{\partial t} = c_v$$

$$l_p \frac{\partial p}{\partial v} = l_v$$

and

$$l_{m_1} - l_{m_2} + l_p \frac{\partial p}{\partial m_1} = l_{W_1} - l_{W_2}$$

Then

$$\eta\,(t, v, m_1) - \eta\,(t_0, v_0, m_{1_0}) = \int_{t_0,\, v_0,\, m_{1_0}}^{t,\, v,\, m_1} \frac{c_v}{\theta}\,\mathrm{d}t + \frac{l_v}{\theta}\,\mathrm{d}v + \frac{l_{W_1} - l_{W_2}}{\theta}\,\mathrm{d}m_1 \tag{2}$$

Hence

$$\frac{\partial \eta}{\partial t} = \frac{c_v}{\theta}, \tag{3}$$

$$\frac{\partial \eta}{\partial v} = \frac{l_v}{\theta}, \tag{4}$$

and

$$\frac{\partial \eta}{\partial m_1} = \frac{l_{W_1} - l_{W_2}}{\theta}. \tag{5}$$

Thus

$$\frac{1}{\theta}\frac{\partial c_v}{\partial v} = \frac{1}{\theta}\frac{\partial l_v}{\partial t} - \frac{l_v}{\theta^2},$$

and therefore

$$\left(\frac{\partial p}{\partial t}\right)_{v,\, m_1} = \frac{l_v}{\theta}; \tag{6}$$

$$\frac{1}{\theta}\frac{\partial c_v}{\partial m_1} = \frac{1}{\theta}\frac{\partial\,(l_{W_1} - l_{W_2})}{\partial t} - \frac{l_{W_1} - l_{W_2}}{\theta^2},$$

and therefore

$$\frac{\partial \mu_1}{\partial t} - \frac{\partial \mu_2}{\partial t} = -\frac{l_{W_1} - l_{W_2}}{\theta}; \tag{7}$$

$$\frac{1}{\theta}\frac{\partial l_v}{\partial m_1} = \frac{1}{\theta}\frac{\partial\,(l_{W_1} - l_{W_2})}{\partial v},$$

and therefore

$$\frac{\partial \mu_1}{\partial v} - \frac{\partial \mu_2}{\partial v} = -\frac{\partial p}{\partial m_1}. \tag{8}$$

36. Relations between unit mass and variable mass binary systems: We shall now prove that if the functions for a binary system of one phase and unit mass are known, the functions for a binary system of one phase and variable mass may be calculated from them alone, without any additional experimental measurements.

We have

$$\frac{l_{m_1}}{\theta} = \left(\frac{\partial \mathbf{n}}{\partial \mathbf{m}_1}\right)_{t,\, p,\, \mathbf{m}_2} = \left[\frac{\partial\,(\mathbf{m}_1 + \mathbf{m}_2)\,\eta}{\partial \mathbf{m}_1}\right]_{t,\, p,\, \mathbf{m}_2} = \eta + (\mathbf{m}_1 + \mathbf{m}_2)\left(\frac{\partial \eta}{\partial \mathbf{m}_1}\right)_{t,\, p,\, \mathbf{m}_2}$$

$$= \eta + m_2\left(\frac{\partial \eta}{\partial m_1}\right)_{t,\, p}$$

$$= \eta + m_2\,\frac{l_{m_1} - l_{m_2}}{\theta};$$

and

$$\frac{l_{m_2}}{\theta} = \left(\frac{\partial \mathbf{n}}{\partial \mathbf{m}_2}\right)_{t,\,p,\,\mathbf{m}_1} = \frac{\partial\,(\mathbf{m}_1 + \mathbf{m}_2)\,\eta}{\partial \mathbf{m}_2} = \eta + (\mathbf{m}_1 + \mathbf{m}_2)\left(\frac{\partial \eta}{\partial \mathbf{m}_2}\right)_{t,\,p,\,\mathbf{m}_1}$$

$$= \eta - m_1\left(\frac{\partial \eta}{\partial m_1}\right)_{t,\,p}$$

$$= \eta - m_1\,\frac{l_{m_1} - l_{m_2}}{\theta}.$$

Also

$$\mu_1 = \left(\frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{m}_1}\right)_{t,\,p,\,\mathbf{m}_2} - l_{m_1} + pv + pm_2\left(\frac{\partial v}{\partial m_1}\right)_{t,\,p}$$

$$= \left(\frac{\partial\,(\mathbf{m}_1 + \mathbf{m}_2)\,\epsilon}{\partial \mathbf{m}_1}\right)_{t,\,p,\,\mathbf{m}_2} - l_{m_1} + pv + pm_2\left(\frac{\partial v}{\partial m_1}\right)_{t,\,p}$$

$$= \epsilon + (\mathbf{m}_1 + \mathbf{m}_2)\left(\frac{\partial \epsilon}{\partial m_1}\right)_{t,\,p} \cdot \frac{\mathbf{m}_2}{(\mathbf{m}_1 + \mathbf{m}_2)^2} - l_{m_1}$$

$$+ pv + pm_2\left(\frac{\partial v}{\partial m_1}\right)_{t,\,p}$$

$$= \epsilon + m_2\left(l_{m_1} - l_{m_2} - p\,\frac{\partial v}{\partial m_1} + \mu_1 - \mu_2\right) - l_{m_1}$$

$$+ pv + pm_2\left(\frac{\partial v}{\partial m_1}\right)_{t,\,p}$$

$$= \epsilon - \theta\eta + pv + m_2\,(\mu_1 - \mu_2)$$

Similarly

$$\mu_2 = \left(\frac{d\boldsymbol{\varepsilon}}{\partial \mathbf{m}_2}\right)_{t,\,p,\,\mathbf{m}_2} - l_{m_2} + pv - pm_1\left(\frac{\partial v}{\partial m_1}\right)_{t,\,p}$$

$$= \epsilon + (\mathbf{m}_1 + \mathbf{m}_2)\left(\frac{\partial \epsilon}{\partial m_1}\right)\left(\frac{-\,\mathbf{m}_1}{(\mathbf{m}_1 + \mathbf{m}_2)^2}\right) - l_{m_2}$$

$$+ pv - pm_1\left(\frac{\partial v}{\partial m_1}\right)_{t,\,p}$$

$$= \epsilon - m_1\left(l_{m_1} - l_{m_2} - p\,\frac{\partial v}{\partial m_1} + \mu_1 - \mu_2\right) - l_{m_2}$$

$$+ pv - pm_1\left(\frac{\partial v}{\partial m_1}\right)_{t,\,p}$$

$$= \epsilon - \theta\eta + pv - m_1\,(\mu_1 - \mu_2)$$

CHAPTER IV

Homogeneous, n-component systems

We shall now consider the general homogeneous system consisting of n components. We assume the system to be variable in mass and composition for different states but homogeneous in each state, i.e. in each state the composition, density, and temperature are the same at all points and the pressure is the same at all points and is the same in all directions at any given point.

37. Definitions of specific volume and mass fraction: The specific volume, v, is defined by the equation

$$v = \frac{\mathbf{v}}{\mathbf{m}_1 + \mathbf{m}_2 + \cdots + \mathbf{m}_n}, \qquad \begin{array}{l} 0 < \mathbf{v} < \infty \\ 0 \leqq \mathbf{m}_k < \infty, 1 \leqq \mathrm{k}\ (\text{integer}) \leqq \mathrm{n} \\ \mathbf{m}_1 + \cdots + \mathbf{m}_n \neq 0, \end{array}$$

where $\mathbf{v}$ denotes the total volume of the system, $\mathbf{m}_k$ the mass of the substance s_k.

We define the mass fraction of the k-th component, m_k, as

$$m_k = \frac{\mathbf{m}_k}{\mathbf{m}_1 + \mathbf{m}_2 + \cdots + \mathbf{m}_n}, \qquad 1 \leqq \mathrm{k} \leqq \mathrm{n}, \text{ k being an integer.}$$

Thus

$$m_1 + \cdots + m_n = 1.$$

38. Equation of state: We assume as a physical hypothesis that

$$\varphi(t, p, v, m_1, m_2, \cdots, m_{n-1}) = 0$$

can be solved explicitly to give any one of the variables as a single-valued continuous function of the others.

Thus

$$\mathbf{v} = (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, v\,(t, p, m_1, \cdots, m_{n-1})$$

We assume further $\mathbf{v} = 0$, $t = t_0$, $p = p_0$, $\mathbf{m}_1 = 0$, $\cdots$, $m_n = 0$ and that

$$\lim_{\mathbf{m}_k \to 0,\ \mathrm{k} = 1, \cdots, \mathrm{n}} \frac{\mathbf{v}}{\mathbf{m}_1 + \cdots + \mathbf{m}_n} = v = 0$$

39. Definitions of work and heat: The work, $\mathbf{W}$, done on the system is defined by the integral

$$\mathbf{W} = -\int_{t_0,\, p_0,\, \mathbf{m}_{1_0},\, \dots,\, \mathbf{m}_{n_0}}^{t,\, p,\, \mathbf{m}_1,\, \dots,\, \mathbf{m}_n} p d\mathbf{v} \quad , \text{ where } \mathbf{v} = (\mathbf{m}_1 + \mathbf{m}_2 + \cdots + \mathbf{m}_n)\, v\,(t, p\, m_1, \cdots, m_{n-1})$$

Expanding we have

$$\mathbf{W} = -\int_{t_0,\, p_0,\, \mathbf{m}_{1_0},\, \dots,\, \mathbf{m}_{n_0}}^{t,\, p,\, \mathbf{m}_1,\, \dots,\, \mathbf{m}_n} (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, p \frac{\partial v}{\partial t} dt + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, p \frac{\partial v}{\partial p} dp + p\left[v + \frac{\partial v}{\partial m_1} - \sum_{i=1}^{n-1} m_i \frac{\partial v}{\partial m_i}\right] d\mathbf{m}_1 + \cdots\cdots + p\left[v - \sum_{i=1}^{n-1} m_i \frac{\partial v}{\partial m_i}\right] d\mathbf{m}_n.$$

In the special case where $\mathbf{m}_1 + \mathbf{m}_2 + \cdots + \mathbf{m}_n = 1$ (total mass constant) $\mathbf{m}_1 = m_1, \cdots, \mathbf{m}_n = m_n$, and the integral reduces to

$$W = -\int_{t_0,\, p_0,\, m_{1_0},\, \dots,\, m_{n-1_0}}^{t,\, p,\, m_1,\, \dots,\, m_{n-1}} p \frac{\partial v}{\partial t} dt + p \frac{\partial v}{\partial p} dp + p \frac{\partial v}{\partial m_1} dm_1 + \cdots + p \frac{\partial v}{\partial m_{n-1}} dm_{n-1}$$

The heat, $\mathbf{Q}$, received by the system is defined by the integral

$$\mathbf{Q} = \int_{t_0,\, p_0,\, \mathbf{m}_{1_0},\, \dots,\, \mathbf{m}_{n_0}}^{t,\, p,\, \mathbf{m}_1,\, \dots,\, \mathbf{m}_n} (\mathbf{m}_1 + \mathbf{m}_2 + \cdots + \mathbf{m}_n)\, [c_p\, dt + l_p\, dp] + l_{m_1} d\mathbf{m}_1 + \cdots + l_{m_n} d\mathbf{m}_n$$

In the special case where $\mathbf{m}_1 + \mathbf{m}_2 + \cdots + \mathbf{m}_n = 1$ (total mass constant) that is $\mathbf{m}_1 = m_1, \cdot\cdot, \mathbf{m}_n = m_n$, this integral reduces to

$$Q = \int_{t_0,\, p_0,\, m_{1_0},\, \dots,\, m_{n-1_0}}^{t,\, p,\, m_1,\, \dots,\, m_{n-1}} c_p\, dt + l_p\, dp + (l_{m_1} - l_{m_n})\, dm_1 + \cdots + (l_{m_{n-1}} - l_{m_n})\, dm_{n-1}$$

40. Transformation of the heat integral: By hypothesis

$$p = p\,(t, v, m_1, \cdots, m_{n-1}) \text{ where } m_k = \frac{\mathrm{m}_k}{\mathrm{m}_1 + \cdots + \mathrm{m}_n}$$

$$1 \leqq k \leqq n - 1,\ k = \text{integer}.$$

Thus

$$\int_{t_0,\, p_0,\, \mathrm{m}_{1_0},\, \ldots,\, \mathrm{m}_{n_0}}^{t,\, p,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n} (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p\, dt + (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, l_p\, dp + l_{m_1}\, d\mathrm{m}_1 + \cdots + l_{m_n}\, d\mathrm{m}_n$$

$$= \int_{t_0,\, v_0,\, \mathrm{m}_{1_0},\, \ldots,\, \mathrm{m}_{n_0}}^{t,\, v,\, \mathrm{m},\, \ldots,\, \mathrm{m}_n} (\mathrm{m}_1 + \cdots + \mathrm{m}_n) \left\{ \left[c_p + l_p \frac{\partial p}{\partial t} \right] dt + \left[l_p \frac{\partial p}{\partial v} \right] dv \right\}$$

$$+ \left[l_{m_1} + l_p \frac{\partial p}{\partial m_1} - l_p \sum_{i=1}^{n-1} m_i \frac{dp}{\partial m_i} \right] d\mathrm{m}_1 + \cdots$$

$$+ \left[l_{m_n} - l_p \sum_{i=1}^{n-1} m_i \frac{\partial p}{\partial m_i} \right] d\mathrm{m}_n.$$

$$= \int_{t_0,\, v_0,\, \mathrm{m}_{1_0},\, \ldots,\, \mathrm{m}_{n_0}}^{t,\, v,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n} (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_v\, dt + (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, l_v\, dv + l_{W_1}\, d\mathrm{m}_1 + \cdots + l_{W_n}\, d\mathrm{m}_n$$

since by definition

$$c_p + l_p \frac{\partial p}{\partial t} = c_v$$

$$l_p \frac{\partial p}{\partial v} = l_v$$

$$l_{mk} + (\Sigma\, \mathrm{m}_a)\, l_p \frac{\partial p}{\partial \mathrm{m}_k} = l_{W_k},\ 1 \leqq k \leqq n,\ k = \text{integer}.$$

In the special case where $\mathrm{m}_1 + \cdots + \mathrm{m}_n = 1$, i. e. in the constant total mass case, this integral reduces to

$$\int_{t,\, v,\, m_{1_0},\, \ldots,\, m_{n-1_0}}^{t,\, v,\, m_1,\, \ldots,\, m_{n-1}} c_v\, dt + l_v\, dv + (l_{W_1} - l_{W_n})\, dm_1 + \cdots + (l_{W_{n-1}} - l_{W_n})\, dm_{n-1}$$

41. The first law of thermodynamics: The first law of thermodynamics is expressed by the equation

$$\varepsilon(t, p, \mathrm{m}_1, \cdots, \mathrm{m}_n) - \varepsilon(t_0, p_0, \mathrm{m}_{1_0}, \cdots, \mathrm{m}_{n_0}) =$$

$$\int_{t_0, p_0, \mathrm{m}_{1_0}, \ldots, \mathrm{m}_{n_0}}^{t, p, \mathrm{m}_1, \ldots, \mathrm{m}_n} \left(\mathrm{m}_1 + \cdots + \mathrm{m}_n\right)\left(c_p - p\frac{\partial v}{\partial t}\right) dt +$$

$$\left(\mathrm{m}_1 + \cdots + \mathrm{m}_n\right)\left(l_p - p\frac{\partial v}{\partial p}\right) dp +$$

$$\left[l_{m_1} - pv - p\frac{\partial v}{\partial m_1} + p\sum_{i=1}^{n-1} m_i \frac{\partial v}{\partial m_i} + \mu_1\right] d\mathrm{m}_1 + \cdots +$$

$$\left[l_{m_n} - pv + p\sum_{i=1}^{n-1} m_i \frac{\partial v}{\partial m_i} + \mu_n\right] d\mathrm{m}_n. \tag{1}$$

where $\mu_1, \cdots, \mu_n$ denote continuous single-valued functions of $t, p, m_1, \cdots, m_{n-1}$.

SPECIAL CASES

(1) In the special case where $\mathrm{m}_1, \cdots, \mathrm{m}_n$ are all constant, the first law reduces to the following:

$$\varepsilon(t, p, \mathrm{m}_1, \cdots, \mathrm{m}_n) - \varepsilon(t_0, p_0, \mathrm{m}_1, \cdots, \mathrm{m}_n) =$$

$$(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\int_{t_0, p_0, \mathrm{m}_1, \ldots, \mathrm{m}_n}^{t, p, \mathrm{m}_1, \ldots, \mathrm{m}_n} \left(c_p - p\frac{\partial v}{\partial t}\right) dt + \left(l_p - p\frac{\partial v}{\partial p}\right) dp \tag{2}$$

Now $\varepsilon(t, p, \mathrm{m}_1, 0, \cdots, 0) = \varepsilon'(t, p, \mathrm{m})$ and

$\varepsilon'(t, p, 0) = 0$ by (23.3).

Hence

$$\varepsilon(t, p, 0, \cdots, 0) = 0 \tag{3}$$

(2) In the special case where $t, p, m_1, \cdots, m_{n-1}$ are constant the first law reduces to

$$\varepsilon(t, p, \mathrm{m}_1, \cdots, \mathrm{m}_n) - \varepsilon(t, p, 0, \cdots, 0) =$$

$$(l_{m_1} - p \quad + \mu_1)\,\mathrm{m}_1 + \cdots + (l_{m_n} - pv + \mu_n)\,\mathrm{m}_n \tag{4}$$

or

$$\boldsymbol{\varepsilon}(t, p, \mathbf{m}_1, \cdots, \mathbf{m}_n) = \mathbf{m}_1 l_{m_1} + \cdots + \mathbf{m}_n l_{m_n} - pv$$
$$(\mathbf{m}_1 + \cdots + \mathbf{m}_n) + \mu_1 \mathbf{m}_1 + \cdots + \mu_n \mathbf{m}_n \qquad (5)$$

Let $\epsilon = \dfrac{\boldsymbol{\varepsilon}}{\mathbf{m}_1 + \cdots + \mathbf{m}_n}$ by definition

Then

$$\epsilon = m_1 l_{m_1} + \cdots + m_n l_{m_n} - pv + \mu_1 m_1 + \cdots + \mu_n m_n \qquad (6)$$
$$= \epsilon(t, p, m_1, \cdots, m_{n-1}).$$

Hence

$$\boldsymbol{\varepsilon}(t, p, \mathbf{m}_1, \cdots, \mathbf{m}_n) = (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, \epsilon(t, p, m_1, \cdots, m_{n-1}) \qquad (7)$$

(3) In the special case where $\mathbf{m}_1 + \mathbf{m}_2 + \cdots + \mathbf{m}_n = 1$, *i.e.* in the constant total mass or unit mass case, $\mathbf{m}_1 = m_1, \cdots, \mathbf{m}_n = m_n$, and the first law reduces to

$$\epsilon(t, p, m_1, \cdots, m_{n-1}) - \epsilon(t_0, p_0, m_{1_0}, \cdots, m_{n-1_0}) =$$

$$\int_{t_0, p_0, m_{1_0}, \ldots, m_{n-1_0}}^{t, p, m_1, \ldots, m_{n-1}} \left(c_p - p\frac{\partial v}{\partial t}\right) dt + \left(l_p - p\frac{\partial v}{\partial p}\right) dp + \Bigg[\Bigg(l_{m_1} - pv - p\frac{\partial v}{\partial m_1}$$
$$+ \sum_{i=1}^{n-1} m_i \frac{\partial v}{\partial m_i} + \mu_1\Bigg) - \left(l_{m_n} - pv + \sum_{i=1}^{n-1} m_i \frac{\partial v}{\partial m_i} + \mu_n\right)\Bigg] dm_1 +$$
$$\cdots + \Bigg[\Bigg(l_{m_{n-1}} - pv - p\frac{\partial v}{\partial m_{n-1}} + \sum_{i=1}^{n-1} m_i \frac{\partial v}{\partial m_i} + \mu_{n-1}\Bigg)$$
$$- \left(l_{m_n} - pv + \sum_{i=1}^{n-1} m_i \frac{\partial v}{\partial m_i} + \mu_n\right)\Bigg] dm_{n-1} \qquad (8)$$

$$= \int_{t_0, p_0, m_{1_0}, \ldots, m_{n-1_0}}^{t, p, m_1, \ldots, m_{n-1}} \left(c_p - p\frac{\partial v}{\partial t}\right) dt + \left(l_p - p\frac{\partial v}{\partial p}\right) dp + \Bigg[l_{m_1} - l_{m_n} - p\frac{\partial v}{\partial m_1}$$
$$+ \mu_1 - \mu_n\Bigg] dm_1 + \cdots\cdots + \Bigg[l_{m_{n-1}} - l_{m_n} - p\frac{\partial v}{\partial m_{n-1}}$$
$$+ \mu_{n-1} - \mu_n\Bigg] dm_{n-1}. \qquad (9)$$

Hence

$$\epsilon_t\,(t, p, m_1, \cdots, m_{n-1}) = c_p - p\left(\frac{\partial v}{\partial t}\right)_{p,\, m_1,\, \ldots,\, m_{n-1}} \tag{10}$$

$$\epsilon_p\,(t, p, m_1, \cdots, m_{n-1}) = l_p - p\left(\frac{\partial v}{\partial p}\right)_{t,\, m_1,\, \ldots,\, m_{n-1}} \tag{11}$$

$$\epsilon_{m_k}\,(t, p, m_1, \cdots, m_{n-1}) = l_{m_k} - l_{m_n} - p\,\frac{\partial v}{\partial m_k} + \mu_k - \mu_n$$

$$\text{where}\quad 1 \leqq k \leqq n-1,\ k = \text{integer} \tag{12}$$

and

$$\epsilon\,(t_0, p_0, m_{1_0}, \cdots, m_{n-1_0}) = \epsilon'\,(t_0, p_0) = 0 \text{ by definition}$$
$$\text{where } m_{k_0} = 1,\ 1 \leqq k \leqq n, \text{ i.e. all } m_{i_0} \text{ zero except } m_{k_0}. \tag{13}$$

Further, from the second derivatives of ϵ, we have

$$\frac{\partial c_p}{\partial p} - \frac{\partial v}{\partial t} = \frac{\partial l_p}{\partial t}, \tag{14}$$

$$\frac{\partial c_p}{\partial m_k} = \frac{\partial l_{m_k}}{\partial t} - \frac{\partial l_{m_n}}{\partial t} + \frac{\partial \mu_k}{\partial t} - \frac{\partial \mu_n}{\partial t}, \tag{15}$$

$$\frac{\partial l_p}{\partial m_k} = \frac{\partial l_{m_k}}{\partial p} - \frac{\partial l_{m_n}}{\partial p} + \frac{\partial \mu_k}{\partial p} - \frac{\partial \mu_n}{\partial p} - \frac{\partial v}{\partial m_k}, \tag{16}$$

$$\text{where}\quad 1 \leqq k \leqq n-1,\ k = \text{integer.}$$

42. Transformation of the energy integral. By hypothesis we have

$$p = p\,(t, v, m_1, \cdots, m_{n-1})$$

Thus

$$\int_{t_0,\, p_0,\, m_{1_0},\, \ldots,\, m_{n_0}}^{t,\, p,\, m_1,\, \ldots,\, m_n} \left(m_1 + \cdots + m_n\right)\left[c_p - p\,\frac{\partial v}{\partial t}\right] dt + \left(m_1 + \cdots + m_n\right)$$

$$\left[l_p - p\,\frac{\partial v}{\partial p}\right] dp + \left[l_{m_1} - pv - p\,\frac{\partial v}{\partial m_1} + p\sum_{i=1}^{n-1} m_i\,\frac{\partial v}{\partial m_i} + \mu_1\right]$$

$$dm_1 + \cdots + \left[l_{m_n} - pv + p\sum_{i=1}^{n-1} m_i\,\frac{\partial v}{\partial m_i} + \mu_n\right] dm_n$$

$$= \int_{t_0, v_0, \mathbf{m}_{1_0}, \ldots\ldots, \mathbf{m}_{n_0}}^{t, v, \mathbf{m}_1, \ldots., \mathbf{m}_n} \left(\mathbf{m}_1 + \cdots + \mathbf{m}_n\right)\left[c_p - p\frac{\partial v}{\partial t} + l_p\frac{\partial p}{\partial t} - p\frac{\partial v}{\partial p}\frac{\partial p}{\partial t}\right] \mathrm{d}t$$

$$+ \left(\mathbf{m}_1 + \cdots + \mathbf{m}_n\right)\left[l_p - p\frac{\partial v}{\partial p}\right]\frac{\partial p}{\partial v}\,\mathrm{d}v$$

$$+ \left[l_{m_1} - pv - p\frac{\partial v}{\partial m_1} + p\sum_{i=1}^{n-1} m_i\frac{\partial v}{\partial m_i} + \mu_1\right.$$

$$\left. + \left(l_p - p\frac{\partial v}{\partial p}\right)\left(\frac{\partial p}{\partial m_1} - \sum_{i=1}^{n-1} m_i\frac{\partial p}{\partial m_i}\right)\right] \mathrm{d}\mathbf{m}_1$$

$$+ \cdots\cdots + \left[l_{m_n} - pv - p\sum_{i=1}^{n-1} m_i\frac{\partial v}{\partial m_i} + \mu_n\right.$$

$$\left. + \left(l_p - p\frac{\partial v}{\partial p}\right)\left(-\sum_{i=1}^{n-1} m_i\frac{\partial p}{\partial m_i}\right)\right] \mathrm{d}\mathbf{m}_n$$

$$= \int_{t_0, v_0, \mathbf{m}_{1_0}, \ldots\ldots, \mathbf{m}_{n_0}}^{t, v, \mathbf{m}_1, \ldots., \mathbf{m}_n} (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_v\, \mathrm{d}t + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,(l_v - p)\,\mathrm{d}v$$

$$+ [l_{W_1} + \mu_1]\,\mathrm{d}\mathbf{m}_1 + \cdots + [l_{W_n} + \mu_n]\,\mathrm{d}\mathbf{m}_n$$

since by definition

$$c_p + l_p\frac{\partial p}{\partial t} = c_v$$

$$l_p\frac{\partial p}{\partial v} = l_v$$

$$l_{m_k} + (\Sigma\,\mathbf{m}_a)\, l_p\frac{\partial p}{\partial \mathbf{m}_k} = l_{W_k},\ 1 \leqq \mathrm{k} \leqq \mathrm{n},\ \mathrm{k} = \text{integer}.$$

In the special case where $\mathbf{m}_1 + \cdots + \mathbf{m}_n = 1$, then $\mathbf{m}_1 = m_1$, $\cdots, \mathbf{m}_n = m_n$, and the integral reduces to

$$\int_{t_0, v_0, m_{1_0}, \ldots.\, m_{n-1_0}}^{t, v, m_1, \ldots\ldots, m_{n-1}} c_v\,\mathrm{d}t + (l_v - p)\,\mathrm{d}v + [l_{W_1} - l_{W_n} + \mu_1 - \mu_n]\,\mathrm{d}m_1$$

$$+ \cdots + [l_{W_{n-1}} - l_{W_n} + \mu_{n-1} - \mu_n]\,\mathrm{d}m_{n-1}$$

where

$$m_n = 1 - m_1 - \cdots - m_{n-1}$$

Hence

$$\epsilon_t\,(t, v, m_1, \cdots, m_{n-1}) = c_v$$

$$\epsilon_v\,(t, v, m_1, \cdots, m_{n-1}) = l_v - p$$

$$\epsilon_{m_k}\,(t, v, m_1, \cdots, m_{n-1}) = l_{W_k} - l_{W_n} + \mu_k - \mu_n,$$

$$\text{where } 1 \leqq k \leqq n-1$$

Thus

$$\frac{\partial c_v}{\partial v} = \frac{\partial l_v}{\partial t} - \frac{\partial p}{\partial t}$$

$$\frac{\partial c_v}{\partial m_k} = \frac{\partial l_{W_k}}{\partial t} - \frac{\partial l_{W_n}}{\partial t} + \frac{\partial \mu_k}{\partial t} - \frac{\partial \mu_n}{\partial t}$$

$$\frac{\partial l_v}{\partial m_k} - \frac{\partial p}{\partial m_k} = \frac{\partial l_{W_k}}{\partial v} - \frac{\partial l_{W_n}}{\partial v} + \frac{\partial \mu_k}{\partial v} - \frac{\partial \mu_n}{\partial v}$$

43. The second law of thermodynamics: The second law of thermodynamics is expressed by the equation

$$\mathfrak{n}\,(t, p, \mathrm{m}_1, \cdots, \mathrm{m}_n) - \mathfrak{n}\,(t_0, p_0, \mathrm{m}_{1_0}, \cdots, \mathrm{m}_{n_0}) =$$

$$\int_{t_0,\, p_0,\, \mathrm{m}_{1_0},\, \ldots,\, \mathrm{m}_{n_0}}^{t,\, p,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n} \left(\mathrm{m}_1 + \cdots + \mathrm{m}_n\right)\frac{c_p}{\theta}\,dt + \left(\mathrm{m}_1 + \cdots + \mathrm{m}_n\right)\frac{l_p}{\theta}\,dp$$

$$+ \frac{l_{m_1}}{\theta}\,d\mathrm{m}_1 + \cdots + \frac{l_{m_n}}{\theta}\,d\mathrm{m}_n$$

where θ is a function of t only, $\theta = \Gamma\,(t)$, the same for all systems.

Now $\mathfrak{n}\,(t, p, \mathrm{m}_1, 0, \cdots, 0) = \mathfrak{n}'\,(t, p, \mathrm{m})$ and $\mathfrak{n}'\,(t, p, 0) = 0$ by (24.3)
Hence $\mathfrak{n}\,(t, p, 0, \cdots, 0) = 0$

SPECIAL CASES

(1) In the special case where $\mathrm{m}_1, \cdots, \mathrm{m}_n$ are all constant the equation reduces to

$$\mathfrak{n}\,(t, p, \mathrm{m}_1, \cdots, \mathrm{m}_n) - \mathfrak{n}\,(t_0, p_0, \mathrm{m}_1, \cdots, \mathrm{m}_n)$$

$$= (\mathrm{m}_1 + \cdots + \mathrm{m}_n) \int_{t_0,\, p_0,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n}^{t,\, p,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n} \frac{c_p}{\theta}\,dt + \frac{l_p}{\theta}\,dp$$

(2) In the special case where $t, p, m_1, \cdots, m_{n-1}$ are all constant

$$\mathbf{n}(t, p, \mathbf{m}_1, \cdots, \mathbf{m}_n) - \mathbf{n}(t, p, 0, \cdots, 0) = \frac{l_{m_1}}{\theta}\mathbf{m}_1 + \cdots + \frac{l_{m_n}}{\theta}\mathbf{m}_n$$

or

$$\mathbf{n}(t, p, \mathbf{m}_1, \cdots, \mathbf{m}_n) = \frac{l_{m_1}}{\theta}\mathbf{m}_1 + \cdots + \frac{l_{m_n}}{\theta}\mathbf{m}_n.$$

Let $\eta = \dfrac{\mathbf{n}}{\mathbf{m}_1 + \cdots + \mathbf{m}_n}$ by definition,

then

$$\eta = \frac{l_{m_1}}{\theta} m_1 + \cdots + \frac{l_{m_n}}{\theta} m_n$$
$$= \eta(t, p, m_1, \cdots, m_{n-1})$$

Therefore from (41.6),

$$\epsilon = m_1 l_{m_1} + \cdots + m_n l_{m_n} - pv + \mu_1 m_1 + \cdots + \mu_n m_n$$

we have

$$\epsilon = \theta\eta - pv + \mu_1 m_1 + \cdots + \mu_n m_n$$

or

$$\boldsymbol{\varepsilon} = \theta\mathbf{n} - p\mathbf{v} + \mu_1 \mathbf{m}_1 + \cdots + \mu_n \mathbf{m}_n$$ [1]

(3) In the special case where $\mathbf{m}_1 + \cdots + \mathbf{m}_n = 1$, *i.e.* where the total mass remains constant, the second law reduces to

$$\eta(t, p\, m_1, \cdots, m_{n-1}) - \eta(t_0, p_0, m_{1_0}, \cdots, m_{n-1_0}) =$$
$$\int_{t_0, p_0, m_{1_0}, \ldots, m_{n-1_0}}^{t, p, m_1, \ldots, m_{n-1}} \frac{c_p}{\theta}\,dt + \frac{l_p}{\theta}\,dp + \frac{l_{m_1} - l_{m_n}}{\theta}\,dm_1 + \cdots + \frac{l_{m_{n-1}} - l_{m_n}}{\theta}\,dm_{n-1}$$

and

$\eta(t_0, p_0, m_1{}'_0, \cdots, m_{n-1}{}'_0) = \eta'(t_0, p_0) = 0$ by definition where $m_k{}'_0 = 1$, $1 \leqq k \leqq n$, $k =$ integer *i.e.* all the $m_i{}'_0$ are zero except $m_k{}'_0$.

[1] This is Gibbs' equation 93.

Hence

$$\eta_t(t, p, m_1, \cdots, m_{n-1}) = \frac{c_p}{\theta}$$

$$\eta_p(t, p, m_1, \cdots\cdot, m_{n-1}) = \frac{l_p}{\theta}$$

$$\eta_{m_k}(t, p, m_1, \cdots\cdot, m_{n-1}) = \frac{l_{m_k} - l_{m_n}}{\theta}, \ 1 \leqq k \leqq n-1$$

Thus, from the second derivatives of η, we have

$$\frac{1}{\theta}\frac{\partial c_p}{\partial p} = \frac{1}{\theta}\frac{\partial l_p}{\partial t} - \frac{l_p}{\theta^2},$$

hence

$$\left(\frac{\partial v}{\partial t}\right)_{p,\, m_1, \ldots, \, m_{n-1}} = -\frac{l_p}{\theta};$$

$$\frac{1}{\theta}\frac{\partial c_p}{\partial m_k} = \frac{1}{\theta}\frac{\partial (l_{m_k} - l_{m_n})}{\partial t} - \frac{l_{m_k} - l_{m_n}}{\theta^2}$$

hence

$$\frac{\partial \mu_k}{\partial t} - \frac{\partial \mu_n}{\partial t} = -\frac{l_{m_k} - l_{m_n}}{\theta}$$

and

$$\frac{1}{\theta}\frac{\partial l_p}{\partial m_k} = \frac{1}{\theta}\frac{\partial (l_{m_k} - l_{m_n})}{\partial p}$$

hence

$$\frac{\partial v}{\partial m_k} = \frac{\partial \mu_k}{\partial p} - \frac{\partial \mu_n}{\partial p}$$

44. Transformation of the entropy integral: By hypothesis we have

$$p = p(t, v, m_1, \cdots\cdot, m_{n-1})$$

Thus

$$\int_{t_0,\, p_0,\, m_{1_0}, \ldots, \, m_{n_0}}^{t,\, p,\, m_1, \ldots, \, m_n} \left(m_1 + \cdots + m_n\right)\frac{c_p}{\theta}\, dt + \left(m_1 + \cdots + m_n\right)\frac{l_p}{\theta}\, dp +$$

$$\frac{l_{m_1}}{\theta}\, dm_1 + \cdots\cdot + \frac{l_{m_n}}{\theta}\, dm_n$$

$$= \int_{t_0,\, v_0,\, m_{1_0},\, \ldots,\, m_{n_0}}^{t,\, v,\, m_1,\, \ldots,\, m_n} \left(m_1 + \cdots + m_n\right)\left[\frac{c_p}{\theta} + \frac{l_p}{\theta}\frac{\partial p}{\partial t}\right] dt +$$

$$\left(m_1 + \cdots + m_n\right)\frac{l_p}{\theta}\frac{\partial p}{\partial v}\, dv + \left[\frac{l_{m_1}}{\theta} + \frac{l_p}{\theta}\frac{\partial p}{\partial m_1} - \frac{l_p}{\theta}\sum_{i=1}^{n-1} m_i \frac{\partial p}{\partial m_i}\right]$$

$$dm_1 + \cdots + \left[\frac{l_{m_n}}{\theta} - \frac{l_p}{\theta}\sum_{i=1}^{n-1} m_i \frac{\partial p}{\partial m_i}\right] dm_n$$

$$= \int_{t_0,\, v_0,\, m_{1_0},\, \ldots,\, m_{n_0}}^{t,\, v,\, m_1,\, \ldots,\, m_n} \left(m_1 + \cdots + m_n\right)\frac{c_v}{\theta}\, dt + \left(m_1 + \cdots + m_n\right)\frac{l_v}{\theta}\, dv$$

$$+ \frac{l_{W_1}}{\theta}\, dm_1 + \cdots + \frac{l_{W_n}}{\theta}\, dm_n$$

In the special case where $m_1 + \cdots + m_n = 1$, *i.e.* where the total mass remains constant the integral reduces to

$$\eta\,(t, v, m_1, \cdots, m_{n-1}) - \eta\,(t_0, v_0, m_{1_0}, \cdots, m_{n-1_0}) =$$

$$\int_{t_0,\, v_0,\, m_{1_0},\, \ldots,\, m_{n-1_0}}^{t,\, v,\, m_1,\, \ldots,\, m_{n-1}} \frac{c_v}{\theta}\, dt + \frac{l_v}{\theta}\, dv + \frac{l_{W_1} - l_{W_n}}{\theta}\, dm_1 + \cdots + \frac{l_{W_{n-1}} - l_{W_n}}{\theta}\, dm_{n-1}$$

Hence

$$\eta_t\,(t, v, m_1, \cdots, m_{n-1}) = \frac{c_v}{\theta}$$

$$\eta_v\,(t, v, m_1, \cdots, m_{n-1}) = \frac{l_v}{\theta}$$

$$\eta_{m_k}\,(t, v, m_1, \cdots, m_{n-1}) = \frac{l_{W_k} - l_{W_n}}{\theta},\ 1 \leqq k \leqq n-1, k = \text{integer}$$

and from the second derivatives of η we have

$$\frac{1}{\theta}\frac{\partial c_v}{\partial v} = \frac{1}{\theta}\frac{\partial l_v}{\partial t} - \frac{l_v}{\theta^2}$$

or

$$\left(\frac{\partial p}{\partial t}\right)_{v,\, m_1,\, \ldots,\, m_{n-1}} = \frac{l_v}{\theta},$$

$$\frac{1}{\theta}\frac{\partial c_v}{\partial m_k} = \frac{1}{\theta}\frac{\partial\,(l_{W_k} - l_{W_n})}{\partial t} - \frac{l_{W_k} - l_{W_n}}{\theta^2}$$

or

$$\frac{\partial \mu_k}{\partial t} - \frac{\partial \mu_n}{\partial t} = - \frac{l_{W_k} - l_{W_n}}{\theta},$$

and

$$\frac{1}{\theta}\frac{\partial l_v}{\partial m_k} = \frac{1}{\theta}\frac{\partial (l_{W_k} - l_{W_n})}{\partial v}$$

or

$$\frac{\partial \mu_k}{\partial v} - \frac{\partial \mu_n}{\partial v} = - \frac{\partial p}{\partial m_k}$$

45. Derivation of Gibbs' equation 12: Assuming that we can solve for θ as a single valued continuous function of $\eta, v, m_1, \cdots\cdot,$ m_{n-1} and obtain ϵ as a continuous function of $\eta, v, m_1, \cdots\cdot, m_{n-1},$

$$\epsilon = \epsilon(\eta, v, m_1, \cdots\cdot, m_{n-1}) \tag{1}$$

we obtain

$$\epsilon_\eta(\eta, v, m_1, \cdots, m_{n-1}) = \left(\frac{\partial \epsilon}{\partial \theta}\right)_{v, m_1, \ldots, m_{n-1}} \left(\frac{\partial \theta}{\partial \eta}\right)_{v, m_1, \ldots, m_{n-1}} = \theta \tag{2}$$

$$\epsilon_v(\eta, v, m_1, \cdots\cdot, m_{n-1}) = \frac{\partial \epsilon}{\partial v} + \frac{\partial \epsilon}{\partial \theta}\frac{\partial \theta}{\partial v} = -p \tag{3}$$

$$\epsilon_{m_k}(\eta, v, m_1, \cdots\cdot, m_{n-1}) = \frac{\partial \epsilon}{\partial m_k} + \frac{\partial \epsilon}{\partial \theta}\frac{\partial \theta}{\partial m_k} \text{ where } 1 \leqq k \leqq n-1$$

$$= \mu_k - \mu_n \tag{4}$$

Thus

$$d\epsilon = \theta\, d\eta - p\, dv + (\mu_1 - \mu_n)\, dm_1 + \cdots\cdot + (\mu_{n-1} - \mu_n)\, dm_{n-1} \tag{5}$$

Now

$$\varepsilon = (\mathrm{m}_1 + \cdots\cdot + \mathrm{m}_n)\, \epsilon(\eta, v, m_1, \cdots\cdot, m_{n-1})$$

where

$$\eta = \frac{\mathrm{n}}{\mathrm{m}_1 + \cdots + \mathrm{m}_n}$$

$$v = \frac{\mathrm{v}}{\mathrm{m}_1 + \cdots + \mathrm{m}_n}$$

$$m_k = \frac{\mathrm{m}_k}{\mathrm{m}_1 + \cdots + \mathrm{m}_n}, \; 1 \leqq k \leqq n-1$$

Thus

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{n}, \mathbf{v}, \mathbf{m}_1, \cdots, \mathbf{m}_n)$$

Hence

$$\boldsymbol{\varepsilon}_{\mathbf{n}}(\mathbf{n}, \mathbf{v}, \mathbf{m}_1, \cdots, \mathbf{m}_n) = (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{\partial \epsilon}{\partial \eta}\frac{\partial \eta}{\partial \mathbf{n}} = \theta$$

$$\boldsymbol{\varepsilon}_{\mathbf{v}}(\mathbf{n}, \mathbf{v}, \mathbf{m}_1, \cdots, \mathbf{m}_n) = (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{\partial \epsilon}{\partial v}\frac{\partial v}{\partial \mathbf{v}} = -p$$

$$\begin{aligned}
\boldsymbol{\varepsilon}_{\mathbf{m}_1}(\mathbf{n}, \mathbf{v}, \mathbf{m}_1, \cdots, \mathbf{m}_n) &= (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\left\{\sum_{i=1}^{n-1}\frac{\partial \epsilon}{\partial m_i}\frac{\partial m_i}{\partial \mathbf{m}_1} + \frac{\partial \epsilon}{\partial \eta}\frac{\partial \eta}{\partial \mathbf{m}_1}\right.\\
&\qquad \left. + \frac{\partial \epsilon}{\partial v}\frac{\partial v}{\partial \mathbf{m}_1}\right\} + \epsilon\\
&= (\mu_1 - \mu_n)(1 - m_1) - (\mu_2 - \mu_n)m_2 - \cdots\\
&\qquad - (\mu_{n-1} - \mu_n)m_{n-1} - \theta\eta + pv + \epsilon\\
&= \mu_1 - \{m_1\mu_1 + \cdots + m_n\mu_n\} - \theta\eta\\
&\qquad + pv + \epsilon\\
&= \mu_1 - (\epsilon + pv - \theta\eta) - \theta\eta + pv + \epsilon\\
&= \mu_1
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\varepsilon}_{\mathbf{m}_n}(\mathbf{n}, \mathbf{v}, \mathbf{m}_1, \cdots, \mathbf{m}_n) &= (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\left\{\sum_{i=1}^{n-1}\frac{\partial \epsilon}{\partial m_i}\frac{\partial m_i}{\partial \mathbf{m}_n} + \frac{\partial \epsilon}{\partial \eta}\frac{\partial \eta}{\partial \mathbf{m}_n}\right.\\
&\qquad \left. + \frac{\partial \epsilon}{\partial v}\frac{\partial v}{\partial \mathbf{m}_n}\right\} + \epsilon\\
&= -(\mu_1 - \mu_n)m_1 - \cdots - (\mu_{n-1}\\
&\qquad - \mu_n)m_{n-1} - \theta\eta + pv + \epsilon\\
&= \mu_n(m_1 + \cdots + m_{n-1}) - \mu_1 m_1 - \cdots\\
&\qquad - \mu_{n-1}m_{n-1} - \theta\eta + pv + \epsilon\\
&= \mu_n(1 - m_n) - \mu_1 m_1 - \cdots - \mu_{n-1}m_{n-1}\\
&\qquad - \theta\eta + pv + \epsilon\\
&= \mu_n
\end{aligned}$$

Thus

$$d\boldsymbol{\varepsilon} = \theta d\mathbf{n} - p d\mathbf{v} + \mu_1 d\mathbf{m}_1 + \cdots + \mu_n d\mathbf{m}_n \quad {}^{(1)}$$

[1] This is Gibbs' equation 12.

46. Definitions of Gibbs' thermodynamic functions: These functions he defines[1] as

$$\zeta = \varepsilon + p\mathbf{v} - \theta\mathbf{n} = \mu_1 \mathbf{m}_1 + \cdots\cdots + \mu_n \mathbf{m}_n$$

$$\chi = \varepsilon + p\mathbf{v} = \theta\mathbf{n} + \mu_1 \mathbf{m}_1 + \cdots\cdots + \mu_n \mathbf{m}_n$$

$$\psi = \varepsilon - \theta\mathbf{n} = -p\mathbf{v} + \mu_1 \mathbf{m}_1 + \cdots\cdots + \mu_n \mathbf{m}_n.$$

Let us define

$$\zeta = \frac{\boldsymbol{\zeta}}{\mathbf{m}_1 + \cdots + \mathbf{m}_n} = \epsilon + pv - \theta\eta$$

$$= \mu_1 m_1 + \cdots + \mu_n m_n$$

$$\chi = \frac{\boldsymbol{\chi}}{\mathbf{m}_1 + \cdots + \mathbf{m}_n} = \epsilon + pv$$

$$= \theta\eta + \mu_1 m_1 + \cdots\cdot + \mu_n m_n$$

$$\psi = \frac{\boldsymbol{\psi}}{\mathbf{m}_1 + \cdots + \mathbf{m}_n} = \epsilon - \theta\eta$$

$$= -pv + \mu_1 m_1 + \cdots\cdot + \mu_n m_n$$

47. Differential and partial derivatives of Gibbs' Zeta: By definition

$$\boldsymbol{\zeta} = \varepsilon(\theta, p, \mathbf{m}_1, \cdots, \mathbf{m}_n) + p\mathbf{v} - \theta\mathbf{n}(\theta, p, \mathbf{m}_1, \cdots, \mathbf{m}_n)$$

$$\text{where } \mathbf{v} = (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, v(\theta, p, m_1, \cdots, m_{n-1})$$

$$= \mathbf{v}(\theta, p, \mathbf{m}_1, \cdots, \mathbf{m}_n)$$

Thus $\boldsymbol{\zeta}$ is a function of $\theta, p, \mathbf{m}_1, \cdots, \mathbf{m}_n$

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}(\theta, p, \mathbf{m}_1, \cdots, \mathbf{m}_n)$$

Hence

$$\boldsymbol{\zeta}_\theta(\theta, p, \mathbf{m}_1, \cdots, \mathbf{m}_n) = \left(\frac{\partial \varepsilon}{\partial \theta}\right)_{p, \mathbf{m}_1, \ldots, \mathbf{m}_n} + p\frac{\partial \mathbf{v}}{\partial \theta} - \mathbf{n} - \theta\frac{\partial \mathbf{n}}{\partial \theta}$$

$$= \left(\mathbf{m}_1 + \cdots + \mathbf{m}_n\right)\left[c_p - p\frac{\partial v}{\partial \theta}\right]$$

$$+ p\left(\mathbf{m}_1 + \cdots + \mathbf{m}_n\right)\frac{\partial v}{\partial \theta} - \mathbf{n}$$

$$- \theta\left(\mathbf{m}_1 + \cdots + \mathbf{m}_n\right)\frac{c_p}{\theta}$$

$$= -\mathbf{n}$$

[1] Gibbs, J. W., Collected Papers, vol. 1, p. 87.

$$\begin{aligned}
\zeta_p(\theta, p, \mathbf{m}_1, \cdots, \mathbf{m}_n) &= \frac{\partial \boldsymbol{\varepsilon}}{\partial p} + \mathbf{v} + p\frac{\partial \mathbf{v}}{\partial p} - \theta\frac{\partial \mathbf{n}}{\partial p} \\
&= \left(\mathbf{m}_1 + \cdots + \mathbf{m}_n\right)\left[l_p - p\frac{\partial v}{\partial p}\right] + \mathbf{v} \\
&\quad + \left(\mathbf{m}_1 + \cdots + \mathbf{m}_n\right) p\frac{\partial v}{\partial p} \\
&\quad - \theta\left(\mathbf{m}_1 + \cdots + \mathbf{m}_n\right)\frac{l_p}{\theta} \\
&= \mathbf{v}
\end{aligned}$$

$$\begin{aligned}
\zeta_{\mathrm{m_k}}(\theta, p, \mathbf{m}_1, \cdots, \mathbf{m}_n) &= \frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{m}_k} + pv + \\
&\quad p\left(\mathbf{m}_1 + \cdots + \mathbf{m}_n\right)\sum_{i=1}^{n-1}\frac{\partial v}{\partial m_i}\frac{\partial m_i}{\partial \mathbf{m}_k} - \theta\frac{\partial \mathbf{n}}{\partial \mathbf{m}_k} \\
&= l_{m_k} - pv - p\frac{\partial v}{\partial m_k} + p\sum_{i=1}^{n-1} m_i\frac{\partial v}{\partial m_i} + \mu_k \\
&\quad + pv + p\frac{\partial v}{\partial m_k} - p\sum_{i=1}^{n-1} m_i\frac{\partial v}{\partial m_i} - \theta\frac{l_{m_k}}{\theta} \\
&= \mu_k \quad , \quad 1 \leqq k \leqq n - 1,\ k = \text{integer}
\end{aligned}$$

$$\begin{aligned}
\zeta_{\mathrm{m_n}}(\theta, p, \mathbf{m}_1, \cdots, \mathbf{m}_n) &= l_{m_n} - pv + p\sum_{i=1}^{n-1} m_i\frac{\partial v}{\partial m_i} + \mu_n + pv - \\
&\quad p\sum_{i=1}^{n-1} m_i\frac{\partial v}{\partial m_i} - \theta\frac{l_{m_n}}{\theta} \\
&= \mu_n
\end{aligned}$$

Thus

$$d\boldsymbol{\zeta} = -\mathbf{n}d\theta + \mathbf{v}dp + \mu_1\, d\mathbf{m}_1 + \cdots + \mu_n\, d\mathbf{m}_n \text{ [1]}$$

In the special case where $\mathbf{m}_1 + \cdots + \mathbf{m}_n = 1$ this equation reduces to

$$d\zeta = -\eta d\theta + vdp + (\mu_1 - \mu_n)\, dm_1 + \cdots + (\mu_{n-1} - \mu_n)\, dm_{n-1}.$$

[1] This is Gibbs' equation 92.

From the second derivatives of ζ we have

$$-\left(\frac{\partial \mathbf{n}}{\partial p}\right)_{\theta,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n} = \left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n}$$

$$-\left(\frac{\partial \mathbf{n}}{\partial \mathrm{m}_k}\right)_{\theta,\, p,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_{k-1},\, \mathrm{m}_{k+1},\, \ldots,\, \mathrm{m}_n} = \left(\frac{\partial \mu_k}{\partial \theta}\right)_{p,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n}$$

$$\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}\right)_{\theta,\, p,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_{k-1},\, \mathrm{m}_{k+1},\, \ldots,\, \mathrm{m}_n} = \left(\frac{\partial \mu_k}{\partial p}\right)_{\theta,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n}$$

or

$$-\left(\frac{\partial \eta}{\partial p}\right)_{\theta,\, m_1,\, \ldots,\, m_{n-1}} = \left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_1,\, \ldots,\, m_{n-1}}$$

$$-\left(\frac{\partial \eta}{\partial m_k}\right)_{\theta,\, p,\, m_1,\, \ldots,\, m_{k-1},\, m_{+-1},\, \ldots,\, m_{n-1}} = \left(\frac{\partial \mu_k}{\partial \theta} - \frac{\partial \mu_n}{\partial \theta}\right)_{p,\, m_1,\, \ldots,\, m_{n-1}}$$

$$\left(\frac{\partial v}{\partial m_k}\right)_{\theta,\, p,\, m_1,\, \ldots,\, m_{k-1},\, m_{k+1},\, \ldots,\, m_{n-1}} = \left(\frac{\partial \mu_k}{\partial p} - \frac{\partial \mu_n}{\partial p}\right)_{\theta,\, m_1,\, \ldots,\, m_{n-1}}$$

48. Differential and partial derivatives of Enthalpy or Gibbs' Chi: By definition

$$\chi = \varepsilon\,(\mathbf{n}, \mathrm{v}, \mathrm{m}_1, \cdots, \mathrm{m}_n) + p\mathrm{v} \text{ where } \mathrm{v} = (\mathrm{m}_1 + \cdots + \mathrm{m}_n)$$

$$v\,(\eta, p, m_1, \cdots, m_{n-1}) \text{ and } \eta = \frac{\mathbf{n}}{\mathrm{m}_1 + \cdots + \mathrm{m}_n}$$

$$= \chi\,(\mathbf{n}, p, \mathrm{m}_1, \cdots, \mathrm{m}_n)$$

Thus

$$\frac{\partial \chi}{\partial \mathbf{n}} = \left(\frac{\partial \varepsilon}{\partial \mathbf{n}}\right)_{\mathrm{v},\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n} + \left(\frac{\partial \varepsilon}{\partial \mathrm{v}}\right)_{\mathbf{n},\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n} \left(\mathrm{m}_1 + \cdots + \mathrm{m}_n\right) \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial \mathbf{n}}$$

$$+ p\left(\mathrm{m}_1 + \cdots + \mathrm{m}_n\right) \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial \mathbf{n}}$$

$$= \theta$$

$$\frac{\partial \chi}{\partial p} = \frac{\partial \varepsilon}{\partial \mathrm{v}} \left(\mathrm{m}_1 + \cdots + \mathrm{m}_n\right) \frac{\partial v}{\partial p} + \mathrm{v} + p\left(\mathrm{m}_1 + \cdots + \mathrm{m}_n\right) \frac{\partial v}{\partial p}$$

$$= \mathrm{v}$$

$$\frac{\partial \chi}{\partial \mathrm{m}_k} = \frac{\partial \varepsilon}{\partial \mathrm{m}_k} + \frac{\partial \varepsilon}{\partial \mathrm{v}} \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k} + p \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}$$

$$= \mu_k$$

Thus

$$d\chi = \theta\, d\mathbf{n} + \mathbf{v}\, dp + \mu_1\, d\mathrm{m}_1 + \cdots + \mu_n\, d\mathrm{m}_n \; {}^{(1)}$$

In the special case where $\mathrm{m}_1 + \cdots + \mathrm{m}_n = 1$ the equation reduces to

$$d\chi = \theta\, d\eta + v\, dp + (\mu_1 - \mu_n)\, dm_1 + \cdots + (\mu_{n-1} - \mu_n)\, dm_{n-1}$$

and

$$\left(\frac{\partial\theta}{\partial p}\right)_{\mathbf{n},\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n} = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{n}}\right)_{p,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n}$$

$$\left(\frac{\partial\theta}{\partial \mathrm{m}_k}\right)_{\mathbf{n},\, p,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_{k-1},\, \mathrm{m}_{k+1},\, \ldots,\, \mathrm{m}_n} = \left(\frac{\partial \mu_k}{\partial \mathbf{n}}\right)_{p,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n}$$

$$\left(\frac{\partial \mathbf{v}}{\partial \mathrm{m}_k}\right)_{\mathbf{n},\, p,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_{k-1},\, \mathrm{m}_{k+1},\, \ldots,\, \mathrm{m}_n} = \left(\frac{\partial \mu_k}{\partial p}\right)_{\mathbf{n},\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n}$$

or

$$\left(\frac{\partial\theta}{\partial p}\right)_{\eta,\, m_1,\, \ldots,\, m_{n-1}} = \left(\frac{\partial v}{\partial \eta}\right)_{p,\, m_1,\, \ldots,\, m_{n-1}}$$

$$\left(\frac{\partial\theta}{\partial m_k}\right)_{\eta,\, p,\, m_1,\, \ldots,\, m_{k-1},\, m_{k+1},\, \ldots,\, m_{n-1}} = \left(\frac{\partial \mu_k}{\partial \eta} - \frac{\partial \mu_n}{\partial \eta}\right)_{p,\, m_1,\, \ldots,\, m_{n-1}}$$

$$\left(\frac{\partial v}{\partial m_k}\right)_{\eta,\, p,\, m_1,\, \ldots,\, m_{k-1},\, m_{k+1},\, \ldots,\, m_{n-1}} = \left(\frac{\partial \mu_k}{\partial p} - \frac{\partial \mu_n}{\partial p}\right)_{\eta,\, m_1,\, \ldots,\, m_{n-1}}$$

49. Differential and partial derivatives of Gibbs' Psi: By definition

$$\psi = \varepsilon(\theta, \mathbf{v}, \mathrm{m}_1, \cdots, \mathrm{m}_n) - \theta\mathbf{n}(\theta, \mathbf{v}, \mathrm{m}_1, \cdots, \mathrm{m}_n)$$
$$= \psi(\theta, \mathbf{v}, \mathrm{m}_1, \cdots, \mathrm{m}_n)$$

Hence

$$\frac{\partial\psi}{\partial\theta} = \left(\frac{\partial\varepsilon}{\partial\theta}\right)_{\mathbf{v},\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n} - \mathbf{n} - \theta\left(\frac{\partial \mathbf{n}}{\partial\theta}\right)_{\mathbf{v},\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n}$$
$$= -\mathbf{n}$$

$$\frac{\partial\psi}{\partial \mathbf{v}} = \left(\frac{\partial\varepsilon}{\partial \mathbf{v}}\right)_{\theta,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n} - \theta\left(\frac{\partial \mathbf{n}}{\partial \mathbf{v}}\right)_{\theta,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n}$$
$$= -p$$

$$\frac{\partial\psi}{\partial \mathrm{m}_k} = \frac{\partial\varepsilon}{\partial \mathrm{m}_k} - \theta\frac{\partial \mathbf{n}}{\partial \mathrm{m}_k}$$
$$= \mu_k$$

[1] This is Gibbs' equation 90.

Thus

$$d\psi = -\mathbf{n}\, d\theta - p\, d\mathbf{v} + \mu_1\, d\mathbf{m}_1 + \cdots + \mu_n\, d\mathbf{m}_n.$$[1]

and

$$d\psi = -\eta\, d\theta - p\, dv + (\mu_1 - \mu_n)\, dm_1 + \cdots + (\mu_{n-1} - \mu_n)\, dm_{n-1}$$

$$\text{where } m_1 + \cdots + m_n = 1.$$

From the second derivatives of ψ we have, where $\theta, \mathbf{v}, \mathbf{m}_1, \cdots, \mathbf{m}_n$ are independent:

$$\frac{\partial \mathbf{n}}{\partial \mathbf{v}} = \frac{\partial p}{\partial \theta}$$

$$-\frac{\partial \mathbf{n}}{\partial \mathbf{m}_k} = \frac{\partial \mu_k}{\partial \theta}$$

$$-\frac{\partial p}{\partial \mathbf{m}_k} = \frac{\partial \mu_k}{\partial \mathbf{v}}$$

or, where $\theta, v, m_1, \cdots, m_{n-1}$ are independent

$$\frac{\partial \eta}{\partial v} = \frac{\partial p}{\partial \theta}$$

$$-\frac{\partial \eta}{\partial m_k} = \frac{\partial \mu_k}{\partial \theta} - \frac{\partial \mu_n}{\partial \theta}$$

$$-\frac{\partial p}{\partial m_k} = \frac{\partial \mu_k}{\partial v} - \frac{\partial \mu_n}{\partial v}$$

50. Derivation of Gibbs' equation 97: Now

$$\epsilon = \theta\eta - pv + \mu_1 m_1 + \cdots\cdots + \mu_n m_n$$

$$= \epsilon\,(\eta, v, m_1, \cdots, m_{n-1}) \tag{1}$$

Let us assume, as a physical hypothesis, that we can solve for $\eta, v,$ and $m_1, \cdots\cdots, m_{n-1}$ in terms of $\theta, \mu_1, \cdots\cdots, \mu_n$.[2]

$$\eta = \eta\,(\theta, \mu_1, \cdots, \mu_n)$$

$$v = v\,(\theta, \mu_1, \cdots, \mu_n) \tag{2}$$

$$m_k = m_k\,(\theta, \mu_1, \cdots, \mu_n),\ 1 \leqq k \leqq n-1,\ k = \text{integer};$$

[1] This is Gibbs' equation 88.

[2] Extensive variables can not be stated in terms of intensive variables alone, *i.e.* the total entropy, volume, and component masses of a system can not be solved for in terms of $\theta, \mu_1, \ldots, \mu_n$. This fact is called attention to here because Gibbs did not state it explicitly in his derivation, to the consequent confusion of many who tried to reproduce his derivation without recognizing this limitation.

thus

$$p = p(\theta, \mu_1, \cdots, \mu_n). \tag{3}$$

Then

$$d\epsilon = \theta\, d\eta + \eta\, d\theta - p\,dv - v\,dp + (\mu_1 - \mu_n)\, dm_1 + m_1\, d\mu_1 + \cdots + m_n\, d\mu_n. \tag{4}$$

where

$$d\eta = \frac{\partial\eta}{\partial\theta} d\theta + \frac{\partial\eta}{\partial\mu_1} d\mu_1 + \cdots + \frac{\partial\eta}{\partial\mu_n} d\mu_n,$$

$$dv = \frac{\partial v}{\partial\theta} d\theta + \frac{\partial v}{\partial\mu_1} d\mu_1 + \cdots + \frac{\partial v}{\partial\mu_n} d\mu_n,$$

$$dp = \frac{\partial p}{\partial\theta} d\theta + \frac{\partial p}{\partial\mu_1} d\mu_1 + \cdots + \frac{\partial p}{\partial\mu_n} d\mu_n,$$

$$dm_k = \frac{\partial m_k}{\partial\theta} d\theta + \frac{\partial m_k}{\partial\mu_1} d\mu_1 + \cdots + \frac{\partial m_k}{\partial\mu_n} d\mu_n.$$

But we already have, from § 45,

$$d\epsilon = \theta\, d\eta - p\,dv + (\mu_1 - \mu_n)\, dm_1 + \cdots + (\mu_{n-1} - \mu_n)\, dm_{n-1} \tag{5}$$

Hence on subtracting (50.4) and (50.5) we obtain

$$0 = \eta\, d\theta - v\, dp + m_1\, d\mu_1 + \cdots\cdots + m_n\, d\mu_n$$

where

$$dp = \frac{\partial p}{\partial\theta} d\theta + \frac{\partial p}{\partial\mu_1} d\mu_1 + \cdots + \frac{\partial p}{\partial\mu_n} d\mu_n.$$

Thus we have

$$dp = \frac{\eta}{v} d\theta + \frac{m_1}{v} d\mu_1 + \cdots\cdots + \frac{m_n}{v} d\mu_n$$

where

$$\frac{\partial p}{\partial\theta} = \frac{\eta}{v}, \frac{\partial p}{\partial\mu_1} = \frac{m_1}{v}, \cdots\cdots, \frac{\partial p}{\partial\mu_n} = \frac{m_n}{v}$$

This equation can also be written, on multiplying numerator and denominator by $\mathbf{m}_1 + \cdots\cdots + \mathbf{m}_n$,

$$dp = \frac{\mathbf{n}}{\mathbf{v}} d\theta + \frac{\mathbf{m}_1}{\mathbf{v}} d\mu_1 + \cdots\cdots + \frac{\mathbf{m}_n}{\mathbf{v}} d\mu_n \text{ [1]}$$

where

$$\frac{\partial p}{\partial\theta} = \frac{\mathbf{n}}{\mathbf{v}}, \frac{\partial p}{\partial\mu_1} = \frac{\mathbf{m}_1}{\mathbf{v}}, \cdots\cdots, \frac{\partial p}{\partial\mu_n} = \frac{\mathbf{m}_n}{\mathbf{v}}.$$

[1] This is Gibbs' equation 97.

CHAPTER V

Strain

So far we have considered only systems for which the work received could be expressed as

$$-\int p\,dv$$

along the path chosen, *i.e.* through the continuous series of equilibrium states in which we are interested.

This is not always true, one important exception being solid systems strained by force fields. Therefore in order to make this discussion more or less complete we should consider systems for which the work received can not be defined in such a simple way. We shall therefore develop the thermodynamic relations for elastically strained solid bodies, first because these systems are very important, and second because the extension of the work definition for these systems is sufficiently general to be readily applicable to systems acted on by other force fields.

"In treating the physical properties of a solid system it is necessary to consider its *state of strain.* A body is said to be *strained* when the relative position of its parts is altered, and by its *state of strain* is meant its state in respect to the relative position of its parts."[1] We have hitherto considered the equilibrium of solids only in the case in which their state of strain is determined by pressures which are the same at all points and the same in all directions at any point. We shall now treat such systems with this limitation removed.

51. Definition of strain: Whenever, owing to any cause, changes take place in the relative positions of the solid system the system is said to be strained.

[1] J. Willard Gibbs. Collected Works, vol. 1, p. 184.

We have, in every such case, to distinguish two states of a body —a first state and a second state. The points of the system pass from their positions in the first state to their positions in the second state by a displacement.

We describe the first state as the state of reference or "unstrained state" and say that so long as the configuration occupied by any part of it remains unchanged in volume and shape, what-

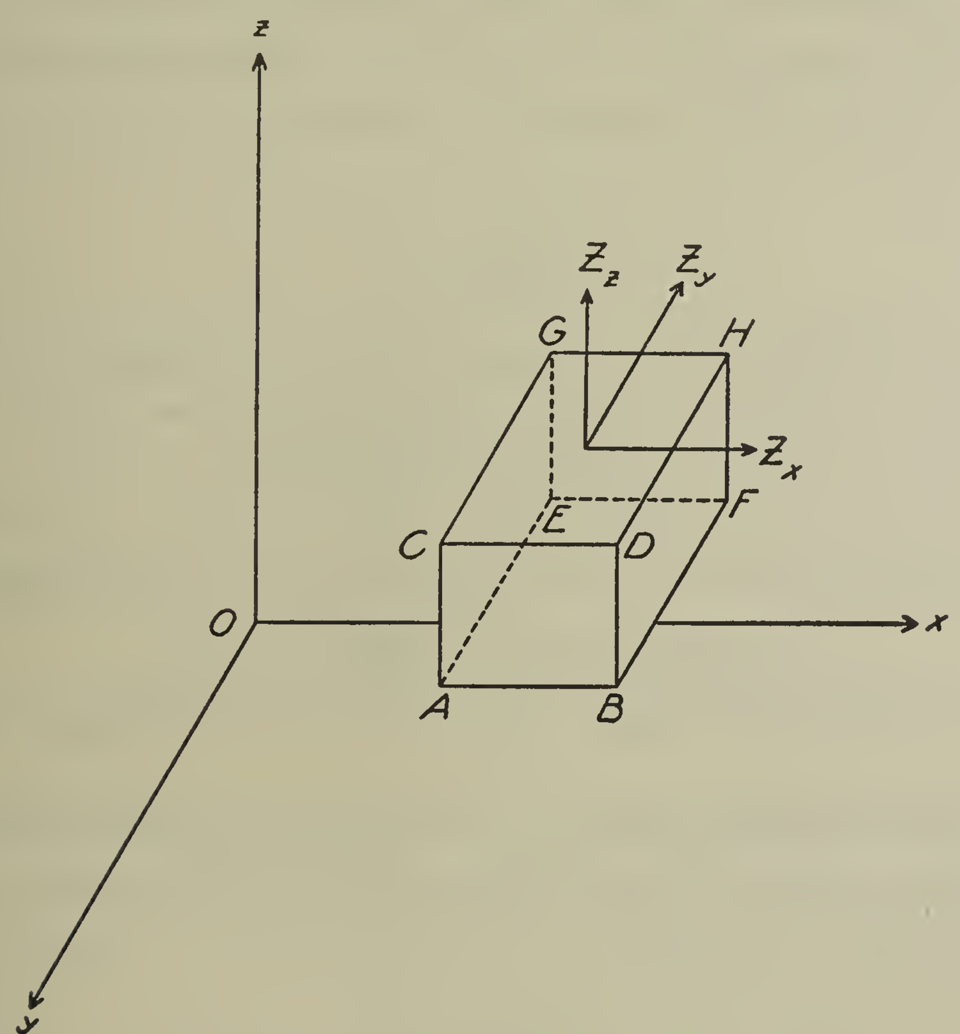

Diagram 3

ever its motion be, the system will be "unstrained," assuming temperature to be constant.

According to the usual notion of strain we may have strains developed by temperature changes with no displacements, *i.e.* the solid is held constant in volume and shape. The strains developed by such temperature changes are equivalent to the strains developed in the body when, after it has been allowed to change freely in volume and shape from the temperature changes, the body is brought back to its original volume and shape by external forces at constant temperature.

If the material in one configuration occupies the volume of, for example, the parallelopiped ABDCEFHG (diagram 3) and if in a second configuration the same material is contained within a volume of different size or shape, we say that the second configuration can be obtained from the first by a process involving strain. Strain may involve a change in the lines, or angles, or both of the parallelopiped.

Let A^1 and B^1 denote the new positions of the points A and B, the change in configuration being a simple displacement along the Ox axis. Then the extension (or contraction) in the length of AB will be

$$\frac{A^1 B^1 - AB}{AB}$$

and the "stretch"[1] e_1 in the direction Ox is denoted by

$$e_1 = \lim_{AB \to 0} \left[\frac{A^1 B^1 - AB}{AB} \right]$$

In much the same way the change in angle CAE is denoted by

$$e_4 = \text{angle CAE} - \text{angle } C^1 A^1 E^1$$

where CAE is a right angle. This type of strain is called "shear."

52. Strain transformations: Let x_0, y_0, z_0 be the coordinates of an arbitrary point P of the volume R at a time s_0, and x_1, y_1, z_1 the coordinates of the point P^1 at a time s_1, $s_1 > s_0$, the displacement being from P to P^1.

Let $x_0 + \Delta x$, $y_0 + \Delta y$, $z_0 + \Delta z$ be the coordinates of a neighboring point Q at a time s_0 and $x_1 + \Delta x_1$, $y_1 + \Delta y_1$, $z_1 + \Delta z_1$ be the coordinates of the point Q^1 at a time s_1, the displacement being from Q to Q^1; i.e. a point in the neighborhood of P transforms into a point in the neighborhood of P^1. Then x_1, y_1, z_1 are uniform and continuous functions of $x_0, y_0, z_0; s_1$ in the volume R.

Thus

$$\begin{aligned} x_1 &= f(x_0, y_0, z_0; s_1) \\ y_1 &= \varphi(x_0, y_0, z_0; s_1) \qquad (1) \\ z_1 &= \psi(x_0, y_0, z_0; s_1) \end{aligned}$$

[1] Love defines this as "extension."

which must satisfy the condition that for $s_1 = s_0$

$$x_1 = x_0,\ y_1 = y_0,\ z_1 = z_0.$$

The relations (52.1) then define a continuous one to one transformation of a region R into a region R_1.

Let P (x, y, z) be the initial positions and $P_1\ (\xi, \eta, \zeta)$ the final positions of the points. If we represent the displacements by vectors we have, designating the projections on the x, y, z axes by u, v, w respectively,

$$\begin{aligned} u &= \xi - x = f\,(x, y, z) - x \\ v &= \eta - y = \varphi\,(x, y, z) - y \\ w &= \zeta - z = \psi\,(x, y, z) - z \end{aligned} \tag{2}$$

Thus u, v, and w are evidently continuous functions of x, y, z in the region for which the transformation is defined. We shall, in general, assume that they are analytic functions in this region.

We assume that the total mass remains constant during this transformation, i.e.

$$\iiint\limits_{V} \rho\, dx\, dy\, dz - \iiint\limits_{V_1} \rho_1\, d\xi\, d\eta\, d\zeta = 0 \tag{3}$$

where ρ denotes the density of the system in the initial state and ρ_1 denotes the density of the system in the final state.

Now since

$$\begin{aligned} \xi &= f\,(x, y, z) \\ \eta &= \varphi\,(x, y, z) \\ \zeta &= \psi\,(x, y, z) \end{aligned} \tag{4}$$

and letting

$$J\,(x, y, z) = \frac{\partial\,(\xi, \eta, \zeta)}{\partial\,(x, y, z)} \neq 0$$

Then

$$\iiint\limits_{V_1} \rho_1\, d\xi\, d\eta\, d\zeta = \iiint\limits_{V} \rho_1 \frac{\partial\,(\xi, \eta, \zeta)}{\partial\,(x, y, z)}\, dx\, dy\, dz \ {}^{(1)} \tag{5}$$

[1] W. F. Osgood, Advanced Calculus, p. 275.

Thus

$$\iiint\limits_{v} (\rho - J\rho_1)\, dx\, dy\, dz = 0 \tag{6}$$

Now this region v was wholly arbitrary. We can understand then by v any sub-region of the original region v and this equation will still hold for this new sub-region. Therefore the integral must vanish at every point of v,

$$\text{or } \rho - J\rho_1 = 0$$

$$\text{or } \rho = J\rho_1 \text{ where } J = \frac{\partial\,(\xi, \eta, \zeta)}{\partial\,(x, y, z)} \tag{7}$$

Let

$$\xi = u + x$$
$$\eta = v + y$$
$$\zeta = w + z;$$

then

$$\begin{array}{lll} \xi_x = u_x + 1, & \xi_y = u_y, & \xi_z = u_z \\ \eta_x = v_x, & \eta_y = v_y + 1, & \eta_z = v_z \\ \zeta_x = w_x, & \zeta_y = w_y, & \zeta_z = w_z + 1 \end{array} \tag{8}$$

and thus

$$J = \begin{vmatrix} u_x + 1 & u_y & u_z \\ v_x & v_y + 1 & v_z \\ w_x & w_y & w_z + 1 \end{vmatrix} \tag{9}$$

53. The functions associated with strain: Let

$$ds^2 = dx^2 + dy^2 + dz^2; \tag{1}$$

and

$$\begin{aligned} ds_1^2 &= d\xi^2 + d\eta^2 + d\zeta^2 \\ &= (\xi_x\, dx + \xi_y\, dy + \xi_z\, dz)^2 + (\eta_x\, dx + \eta_y\, dy + \\ &\quad \eta_z\, dz)^2 + (\zeta_x\, dx + \zeta_y\, dy + \zeta_z\, dz)^2 \\ &= (\xi^2_x + \eta^2_x + \zeta^2_x)\, dx^2 + (\xi^2_y + \eta^2_y + \zeta^2_y)\, dy^2 + \\ &\quad (\xi^2_z + \eta^2_z + \zeta^2_z)\, dz^2 + 2\,(\xi_x\, \xi_y + \eta_x\, \eta_y + \zeta_x\, \zeta_y)\, dx\, dy + \\ &\quad 2\,(\xi_x\, \xi_z + \eta_x\, \eta_z + \zeta_x\, \zeta_z)\, dx\, dz + 2\,(\xi_y\, \xi_z + \\ &\quad \eta_y\, \eta_z + \zeta_y\, \zeta_z)\, dy\, dz \end{aligned} \tag{2}$$

or

$$= [(1 + u_x)^2 + v^2_x + w^2_x]\, dx^2 + [u^2_y + (1 + v_y)^2 + w^2_y]\, dy^2 + [u^2_z + v^2_z + (1 + w_z)^2]\, dz^2 + 2\,[(1 + u_x)\, u_y + (1 + v_y)\, v_x + w_x w_y]\, dx\, dy + 2\,[(1 + u_x)\, u_z + v_x v_z + (1 + w_z)\, w_x]\, dx\, dz + 2\,[u_y u_z + (1 + v_y)\, v_z + (1 + w_z)\, w_y]\, dy\, dz \tag{3}$$

Following the usual notation in elasticity we have

$$ds_1^2 = (1 + 2\,e_1)\, dx^2 + (1 + 2\,e_2)\, dy^2 + (1 + 2\,e_3)\, dz^2 + 2\,e_4\, dy\, dz + 2\,e_5\, dx\, dz + 2\,e_6\, dx\, dy \tag{4}$$

where

$$\begin{aligned} e_1 &= u_x + \frac{1}{2}\left[u^2_x + v^2_x + w^2_x\right] \\ e_2 &= v_y + \frac{1}{2}\left[u^2_y + v^2_y + w^2_y\right] \\ e_3 &= w_z + \frac{1}{2}\left[u^2_z + v^2_z + w^2_z\right] \\ e_4 &= w_y + v_z + u_y u_z + v_y v_z + w_y w_z \\ e_5 &= w_x + u_z + u_x u_z + v_x v_z + w_x w_z \\ e_6 &= v_x + u_y + u_x u_y + v_x v_y + w_x w_y \end{aligned} \tag{5}$$

Thus $e_1, \cdots, e_6$ are continuous functions of x, y, z, and are called the COMPONENTS OF STRAIN. The strain at any point is entirely determined by these six quantities.

54. A physical interpretation of the quantities e_1, e_2, e_3: Divide equation (53.4) by ds^2, and since

$$\frac{dx^2 + dy^2 + dz^2}{ds^2} = 1$$

and the extension or stretch of the vector is given by the expression

$$\lim_{\Delta s \to 0} \frac{\Delta s_1 - \Delta s}{\Delta s} = \frac{ds_1}{ds} - 1,$$

thus the extensions or stretch of the components, which are parallel to the axes of the coordinates in the unstrained state, are respectively

$$\begin{aligned}&\sqrt{1+2e_1}-1\\&\sqrt{1+2e_2}-1\\&\sqrt{1+2e_3}-1\end{aligned}\tag{1}$$

where the positive values of the square roots are taken. We thus obtain a physical interpretation of the quantities e_1, e_2, e_3.

55. The angle between two curves altered by strain. A physical interpretation of the quantities e_4, e_5, e_6: Let l, m, n, and l^1, m^1, n^1 be the direction cosines of two vectors, issuing from the point (x, y, z) in the unstrained state and let θ be the angle between them. Let l_1, m_1, n_1 and $l_1{}^1$, $m_1{}^1$, $n_1{}^1$ be the direction cosines of the corresponding lines in the strained state and θ_1 the angle between them.

Thus $l = x_s$, $m = y_s$, $n = z_s$.

$$l_1 = \frac{d\,(u+x)}{ds_1},\ m_1 = \frac{d\,(y+v)}{ds_1},\ n_1 = \frac{d\,(z+w)}{ds_1}\tag{1}$$

then

$$\begin{aligned}l_1 &= \frac{dx}{ds}\frac{ds}{ds_1} + \frac{\partial u}{\partial x}\frac{dx}{ds}\frac{ds}{ds_1} + \frac{\partial u}{\partial y}\frac{dy}{ds}\frac{ds}{ds_1} + \frac{\partial u}{\partial z}\frac{dz}{ds}\frac{ds}{ds_1}\\ &= \frac{ds}{ds_1}\Big[l\,(1+u_x) + mu_y + nu_z\Big]\end{aligned}\tag{2}$$

similarly for m_1, n_1.

Now

$$\begin{aligned}\cos\theta_1 &= l_1\,l_1{}^1 + m_1\,m_1{}^1 + n_1\,n_1{}^1\\ &= \frac{ds}{ds_1}\frac{ds^1}{ds_1{}^1}\Big[l\,(1+u_x) + mu_y + nu_z\Big]\\ &\qquad \Big[l^1\,(1+u^1{}_x) + \cdots\Big] + \cdots\cdots\end{aligned}$$

$$= \frac{ds}{ds_1}\frac{ds^1}{ds_1{}^1}\Big[ll^1 + ll^1 u_x + \cdots + mm^1 + \cdots\cdots +$$

$$nn^1 + \cdots\cdots\cdots\Big];$$

substituting

$$\cos\theta = ll^1 + mm^1 + nn^1, \text{ and the values } e_1, e_2, \cdots\cdots, e_6$$

in the above expression we have

$$\cos\theta_1 = \frac{ds}{ds_1}\frac{ds^1}{ds_1{}^1}\Big[\cos\theta + 2\,(e_1\,ll^1 + e_2\,mm^1 + e_3\,nn^1) +$$

$$e_4\,(mn^1 + m^1n) + e_5\,(nl^1 + n^1l) + e_6\,(lm^1 + l^1m)\Big]$$

Now let the two given directions be the positive directions of the axes of y and z.

Thus $l = l^1 = 0$, and either $m = n^1 = 0$, $m^1 = n = 1$, or $m^1 = n = 0$, $m = n^1 = 1$.

Let us arbitrarily choose $m = n^1 = 1$

then

$$\cos\theta_1 = \frac{ds}{ds_1}\frac{ds^1}{ds_1{}^1}\Big[1 + e_4\,(1 + 0)\Big]$$

or

$$e_4 = \cos\theta_1 \frac{ds_1}{ds}\frac{ds_1{}^1}{ds^1}$$

and since

$$\left(\frac{ds_1}{ds}\right)^2 = 0 + (1 + 2\,e_2) + 0$$

$$\left(\frac{ds_1{}^1}{ds^1}\right)^2 = 0 + 0 + (1 + 2\,e_3) + 0$$

thus

$$e_4 = \cos\theta_1\sqrt{(1 + 2\,e_2)\,(1 + 2\,e_3)}$$

which gives us an interpretation of the quantity e_4. Similarly with e_5, e_6.

56. Linear dilatation at a point of the system; ELLIPSOID OF DILATATION at this point: Let the displacement be from P to P_1. Let P^1 be a point in the neighborhood of P, and $P_1{}^1$ the displacement of this point. From our assumption of continuity $P_1{}^1$ will be in the neighborhood of P_1.

In the strain $PP^1 = \Delta s$ is displaced to $P_1 P_1{}^1 = \Delta s_1$ which is, in general, a change in length and orientation. Let us consider changes of length only,

$$\delta = \lim_{PP^1 \to 0} \frac{P_1 P_1{}^1 - PP^1}{PP^1} = \frac{ds_1}{ds} - 1 \tag{1}$$

where

$$\delta \gtreqless 0, \ \delta \neq -1.$$

Let l, m, n, be the direction cosines of PP^1, *i.e.*

$$l = \frac{dx}{ds}, \text{ etc.} \tag{2}$$

Now from equation (53.4) we have

$$ds_1{}^2 = (1 + 2\,e_1)\,dx^2 + (1 + 2\,e_2)\,dy^2 + \cdots\cdots\cdots\cdots$$

therefore

$$\left(\frac{ds_1}{ds}\right)^2 = (1 + 2\,e_1)\,l^2 + (1 + 2\,e_2)\,m^2 + (1 + 2\,e_3)\,n^2 + 2\,e_4\,mn + 2\,e_5\,nl + 2\,e_6\,lm. \tag{3}$$

We want to represent the change in $\frac{ds_1}{ds}$ about a point P. To do this let us take on PP^1 a length PQ where

$$PQ = \frac{ds}{ds_1} = \frac{1}{1 + \delta},$$

and let the vector PQ revolve an angle of $2\,\pi$ about P as center.

Let Px, Py, Pz represent the system of rectangular coordinates. Thus in the unstrained state we have

$$x^2 + y^2 + z^2 = 1 \tag{4}$$

and

$$x = l\,(PQ) = l\,\frac{ds}{ds_1}$$

$$y = m\,(PQ) = m\,\frac{ds}{ds_1}$$

$$z = n\,(PQ) = n\,\frac{ds}{ds_1};$$

then the equation (56.3) reduces to

$$1 = \left(\frac{ds_1}{ds}\right)^2\left(\frac{ds}{ds_1}\right)^2 = (1 + 2\,e_1)\,x^2 + (1 + 2\,e_2)\,y^2 + (1 + 2\,e_3)\,z^2 + 2\,e_4\,yz + 2\,e_5\,zx + 2\,e_6\,xy = \psi\,(x, y, z) \qquad (5)$$

and this quadratic is an ellipsoid about P as center.

The points of no change must satisfy equations (56.4) and (56.5) simultaneously (assuming that the sphere touches or cuts the ellipsoid, otherwise no such points exist).

Thus

$$\begin{array}{l}
1 = (1+2\,e_1)\,x^2 + (1+2\,e_2)\,y^2 + (1+2\,e_3)\,z^2 + 2\,e_4\,yz + 2\,e_5\,zx + 2\,e_6\,xy \\
1 = \qquad x^2 + \qquad y^2 + \qquad z^2 \\
\hline
0 = \quad e_1\,x^2 + \quad e_2\,y^2 + \quad e_3\,z^2 + e_4\,yz + e_5\,zx + e_6\,xy
\end{array} \qquad (6)$$

57. Homogeneous strain: Assume now that the deformation is the same at each point P, *i.e.* the case where the six functions e_i do not vary with the position of P in the system.

We define, after Sir W. Thomson, the case where these six quantities are constant with respect to x, y, z, as HOMOGENEOUS STRAIN.

The ellipsoids of dilatation at each point P are then all equal and oriented in the same fashion. Moreover the linear dilatation of an element of the initial system depends only on its direction and not on its position in this system; thus two parallel straight lines are transformed into two parallel straight lines. Two non-parallel straight lines will in general have an angular dilatation as well.

Let the point P (x, y, z) be displaced to the point P (ξ, η, ζ). Now let y, z be constant and x vary.

From equations (53.4) and (53.2) we have

$$1 + 2\,e_1, \text{ constant by hypothesis, } = \left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial x}\right)^2$$

Now the cosine directions l, m, n of the tangent at P (ξ, η, ζ) are constants by hypothesis, thus

$$l = \text{const. by hyp.} = \frac{\dfrac{\partial \xi}{\partial x}}{\sqrt{\left(\dfrac{\partial \xi}{\partial x}\right)^2 + \left(\dfrac{\partial \eta}{\partial x}\right)^2 + \left(\dfrac{\partial \zeta}{\partial x}\right)^2}} = \frac{\dfrac{\partial \xi}{\partial x}}{\sqrt{1 + 2\,e_1}} \tag{1}$$

similarly

$$m = \frac{\eta_x}{\sqrt{1 + 2\,e_1}}, \quad n = \frac{\zeta_x}{\sqrt{1 + 2\,e_1}}, \tag{2}$$

Therefore ξ_x, η_x, ζ_x are constants, *i.e.* ξ, η, ζ are linear functions of x. Similarly ξ, η, ζ are linear functions of y and of z. Finally ξ, η, ζ are linear functions of x, y, z.

Hence we can write the transformation defining a homogeneous strain in the form

$$\begin{aligned} \xi &= a_{10} + (1 + a_{11})\,x + a_{12}\,y + a_{13}\,z \\ \eta &= a_{20} + a_{21}\,x + (1 + a_{22})\,y + a_{23}\,z \\ \zeta &= a_{30} + a_{31}\,x + a_{32}\,y + (1 + a_{33})\,z \end{aligned} \tag{3}$$

the coefficients a_{ij} being constants, and the equations subject to the condition that they give, inversely, for each system of values of ξ, η, ζ a single system of values for x, y, z, *i.e.* that

$$J = \begin{vmatrix} 1 + a_{11} & a_{12} & a_{13} \\ a_{21} & 1 + a_{22} & a_{23} \\ a_{31} & a_{32} & 1 + a_{33} \end{vmatrix} \neq 0$$

From a physical analysis of the situation $J > 0$.

Then

$$
\begin{aligned}
e_1 &= a_{11} + \frac{1}{2}\left[a_{11}^2 + a_{21}^2 + a_{31}^2\right] \\
e_2 &= a_{22} + \frac{1}{2}\left[a_{12}^2 + a_{22}^2 + a_{32}^2\right] \\
e_3 &= a_{33} + \frac{1}{2}\left[a_{13}^2 + a_{23}^2 + a_{33}^2\right] \qquad (4) \\
e_4 &= a_{32} + a_{23} + a_{12}\,a_{13} + a_{22}\,a_{23} + a_{32}\,a_{33} \\
e_5 &= a_{31} + a_{13} + a_{11}\,a_{13} + a_{21}\,a_{23} + a_{31}\,a_{33} \\
e_6 &= a_{21} + a_{12} + a_{11}\,a_{12} + a_{21}\,a_{22} + a_{31}\,a_{32}
\end{aligned}
$$

Let us simplify these formulae by the aid of a translation. Since a_{10}, a_{20}, a_{30} do not enter into e_i they can be made zero without modifying the deformation.

Let

$$
\begin{aligned}
\xi &= \xi' + a_{10} \\
\eta &= \eta' + a_{20} \\
\zeta &= \zeta' + a_{30}
\end{aligned}
$$

then

$$
\begin{aligned}
\xi' &= (1 + a_{11})\,x + a_{12}\,y + a_{13}\,z \\
\eta' &= a_{21}\,x + (1 + a_{22})\,y + a_{23}\,z \qquad (5) \\
\zeta' &= a_{31}\,x + a_{32}\,y + (1 + a_{33})\,z
\end{aligned}
$$

In these formulae the origin goes into itself since when $0 = x = y = z$ we have $\xi' = \eta' = \zeta' = 0$.

In the following treatment we shall assume that a_{10}, a_{20}, a_{30} have been eliminated by the above transformation and drop the primes.

58. Pure strain: In homogeneous strain there is one set of three orthogonal lines in the unstrained state which remain orthogonal after the strain, the direction of these lines being in general altered by the strain.

A homogeneous strain is defined as a PURE STRAIN when there exists in the first state three orthogonal lines which remain

unaltered in direction by the strain. These directions are called the principal directions of the pure strain.

Let OXYZ be the principal directions.

Let x, y, z be the coordinates of the point P before deformation, and ξ, η, ζ be the coordinates of the corresponding point P_1 after deformation.

The deformation being homogeneous, the expressions for ξ, η, ζ as functions of x, y, z are given by

$$\begin{aligned}
\xi &= (1 + a_{11})\, x + a_{12}\, y + a_{13}\, z \\
\eta &= a_{21}\, x + (1 + a_{22})\, y + a_{23}\, z \\
\zeta &= a_{31}\, x + a_{32}\, y + (1 + a_{33})\, z
\end{aligned} \tag{1}$$

where a_{ij} are constants.

By hypothesis, when P is on OX then P_1 is on OX; thus, when y and z are zero, η and ζ must be zero whatever value x may have, *i.e.* a_{21} and a_{31} must be zero identically. Similarly all the other coefficients except a_{11}, a_{22}, a_{33} are zero.

Thus we have the pure deformation referred to its principal directions given by

$$\begin{aligned}
\xi &= (1 + a_{11})\, x \\
\eta &= (1 + a_{22})\, y \\
\zeta &= (1 + a_{33})\, z.
\end{aligned} \tag{2}$$

Now in order that a homogeneous deformation be a pure deformation it is necessary and sufficient that the field of vectors PP_1 be derived from a function of the vectors, *i.e.* that u, v, w be the derivatives of a function of second degree in x, y, z.

The condition indicated is necessary for, if the deformation is pure, then on taking for axes the principal directions OXYZ we have the coordinates of P_1 defined as functions of those of P by

$$\begin{aligned}
\xi &= (1 + a_{11})\, x \\
\eta &= (1 + a_{22})\, y \\
\zeta &= (1 + a_{33})\, z,
\end{aligned}$$

from which, for the projections u, v, w of the vector PP_1 on the axes OXYZ, we have

$$u' = \xi - x = a_{11}\, x,\ v' = a_{22}\, y,\ w' = a_{33}\, z$$

which one can write

$$u' = \frac{\partial F}{\partial x}, \; v' = \frac{\partial F}{\partial y}, \; w' = \frac{\partial F}{\partial z},$$

on setting

$$F = \frac{1}{2}\left[a_{11}\, x^2 + a_{22}\, y^2 + a_{33}\, z^2\right].$$

It is sufficient: for suppose that, in the general formulae defining u, v, w for any homogeneous deformation with respect to the axes OXYZ, u, v, w be the partial derivatives of a similar function F(x, y, z), or, what amounts to the same thing, suppose that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}, \dots\dots\dots,$$

i.e.

$$a_{12} = a_{21}, \; a_{23} = a_{32}, \; a_{31} = a_{13}.$$

The function F is then

$$F = \frac{1}{2}\left[a_{11}\, x^2 + a_{22}\, y^2 + a_{33}\, z^2 + 2\, a_{23}\, yz + 2\, a_{31}\, zx + 2\, a_{12}\, xy\right],$$

which gives

$$u = \frac{\partial F}{\partial x}, \; v = \frac{\partial F}{\partial y}, \; w = \frac{\partial F}{\partial z}.$$

The surfaces of second degree given by

$$F(x, y, z) = \text{const.}$$

play with respect to the vector u, v, w the role of plane surfaces. These surfaces have the same center O and the same directions of the principal axes OXYZ. Let us revolve the axes so that we have as our new axes OXYZ and call x′, y′, z′ the new coordinates of a point P, and u′, v′, w′ the new projections of the vector PP_1; the function F will take on the reduced form

$$F = \frac{1}{2}\left[a_{11}\, x'^2 + a_{22}\, y'^2 + a_{33}\, z'^2\right],$$

which gives

$$u' = \frac{\partial F}{\partial x'} = a_{11}\, x',$$

$$v' = \frac{\partial F}{\partial y'} = a_{22}\, y',$$

$$w' = \frac{\partial F}{\partial z'} = a_{33}\, z'.$$

The deformation is then pure, being equivalent to three simple extensions in three directions mutually at right angles.

59. Strain tangent at a point: Let P be any point in the system before the strain and have as coordinates x, y, z; and let ρ represent an infinitesimal portion of the system around this point P, *i.e.*, let ρ represent the neighborhood of P. Let P^1 $(x + \Delta x,\ y + \Delta y,\ z + \Delta z)$ be a point in ρ.

Let P be displaced to P_1 in the strain, then the coordinates of P_1 will be $\xi = x + u$, $\eta = y + v$, $\zeta = z + w$, and if P^1 is displaced to $P_1{}^1$ in the strain then the coordinates of $P_1{}^1$ are given by

$$\begin{aligned} \xi + \Delta\xi &= x + u + \Delta\,(x + u) \\ \eta + \Delta\eta &= y + v + \Delta\,(y + v) \\ \zeta + \Delta\zeta &= z + w + \Delta\,(z + w) \end{aligned} \qquad (1)$$

which, on expanding $\Delta\ (x + u)$, etc., can be written in the form

$$\begin{aligned} &x + u + \Delta x + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z \\ &y + v + \Delta y + \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \frac{\partial v}{\partial z}\Delta z \\ &z + w + \Delta z + \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z \end{aligned} \qquad (2)$$

when squares and products of Δx, Δy, Δz are neglected. (When the displacement is sufficiently small the approximation involved in this simplification will lie within the experimental error and thus will give an accurate expression of the physical facts.)

Now let us change our coordinates so that P(x, y, z) is the origin and Px^1, Py^1, Pz^1 the coordinate axes.

Let $P^1(x^1, y^1, z^1)$ and $P_1^1(\xi^1, \eta^1, \zeta^1)$ be the coordinates of P^1 and P_1^1 respectively with respect to this coordinate system; we then have the coordinates of P^1 given as $x^1 = \Delta x$, $y^1 = \Delta y$, $z^1 = \Delta z$, and those of P_1^1 given by

$$\begin{aligned} \xi^1 &= u + (1 + u_x)\, x^1 + u_y\, y^1 + u_z\, z^1 \\ \eta^1 &= v + v_x\, x^1 + (1 + v_y)\, y^1 + v_z\, z^1 \\ \zeta^1 &= w + w_x\, x^1 + w_y\, y^1 + (1 + w_z)\, z^1 \end{aligned} \tag{3}$$

Formulae (59.3) giving the coordinates of the point P_1^1 as functions of the coordinates of P^1 define the deformation of the region m around P and m_1 around P_1. As they are linear in x^1, y^1, z^1 they define a homogeneous deformation, and we can accordingly say that, in a sufficiently small neighborhood of a point of the system, the relative displacements are linear functions of the coordinates, i.e. THE STRAIN ABOUT ANY POINT IS SENSIBLY HOMOGENEOUS.

In identifying with formulae (58.1) it is necessary to make

$$a_{11} = u_x,\ a_{12} = u_y,\ a_{13} = u_z,\ a_{21} = v_x,\ a_{22} = v_y,\ a_{23} = v_z,$$
$$a_{31} = w_x,\ a_{32} = w_y,\ a_{33} = w_z. \tag{4}$$

In order that the homogeneous deformation defined by (59.3) be a pure deformation it is necessary and sufficient that $a_{32} = a_{23}$, $a_{13} = a_{31}$, $a_{12} = a_{21}$ (see theorem back) *i.e.* that $w_y - v_z = 0$, $u_z - w_x = 0$, $v_x - u_y = 0$.

60. Very small deformations: Suppose that a continuous medium is deformed in a continuous manner and so that the displacement PP_1 of each of its points P be sufficiently small such that the scalar squares and products of these displacements can be neglected, *i.e.* the approximation involved would lie within the experimental error. Such a strain has also been called infinitely small.

The six characteristic functions referred to the general equations (53.5) neglecting the squares of the partial derivatives of u, v, w with respect to x, y, z give us

$$e_1 = u_x,\ e_2 = v_y,\ e_3 = w_z,\ e_4 = w_y + v_z,\ e_5 = w_x + u_z,$$
$$e_6 = v_x + u_y. \tag{1}$$

Then u, v, w can be written

$$
\begin{aligned}
u &= u_x x + u_y y + u_z z \\
&= e_1 x + \frac{1}{2}(e_6 - v_x + u_y)\, y + \frac{1}{2}(e_5 - w_x + u_z)\, z \\
&= e_1 x + \frac{1}{2} e_6 y + \frac{1}{2} e_5 z - \omega_z y + \omega_y z \qquad (2)
\end{aligned}
$$

similar y

$$v = \frac{1}{2} e_6 x + e_2 y + \frac{1}{2} e_4 z - \omega_x z + \omega_z x \qquad (3)$$

$$w = \frac{1}{2} e_5 x + \frac{1}{2} e_4 y + e_3 z - \omega_y x + \omega_x y \qquad (4)$$

where

$$2\,\omega_x = w_y - v_z,\ 2\,\omega_y = u_z - w_x,\ 2\,\omega_z = v_x - u_y.$$

61. Conditions of compatibility for strain-components: The fact that all six of the strain-components can be expressed in terms of the three component displacements indicates that these six quantities must be, to some extent, interconnected, *i.e.* if we assign an arbitrary expression to each strain component we shall not in general obtain a possible distribution of strain since the conditions for continuity of the system after strain will as a rule be violated.

The values of the components of strain e_i as functions of x, y, z must satisfy the relations $e_1 = u_x$, etc. If we introduce the three relations $2\,\omega_x = w_y - v_z$, etc., then all the partials of u, v, w with respect to x, y, z can be expressed in terms of $e_1, \cdots\cdots, e_6, \omega_x, \omega_y, \omega_z$.

We have

$$
\begin{aligned}
u_y &= u_y + \frac{1}{2} v_x - \frac{1}{2} v_x \\
&= \frac{1}{2}(v_x + u_y) - \frac{1}{2}(v_x - u_y) \\
&= \frac{1}{2} e_6 - \omega_z \qquad (1)
\end{aligned}
$$

$$\begin{aligned} v_x &= v_x + \frac{1}{2}u_y - \frac{1}{2}u_y \\ &= \frac{1}{2}(v_x + u_y) + \frac{1}{2}(v_x - u_y) \\ &= \frac{1}{2}e_6 + \omega_z \end{aligned} \tag{2}$$

Similarly for u_z, w_x, w_y, v_z.

The conditions that these be compatible with the equations

$$u_x = e_1, \cdot\cdot\cdot\cdot\cdot$$

are given by the 9 equations

$$\frac{\partial e_1}{\partial y} = \frac{1}{2}\frac{\partial e_6}{\partial x} - \frac{\partial \omega_z}{\partial x}, \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot$$

which express the partials of ω_x, ω_y, ω_z, in terms of e_i.

Now the equations containing ω_x are

$$2\frac{\partial \omega_x}{\partial x} = \frac{\partial e_5}{\partial y} - \frac{\partial e_6}{\partial z} \tag{3}$$

$$2\frac{\partial \omega_x}{\partial y} = \frac{\partial e_4}{\partial y} - 2\frac{\partial e_2}{\partial z} \tag{4}$$

$$2\frac{\partial \omega_x}{\partial z} = 2\frac{\partial e_3}{\partial y} - \frac{\partial e_4}{\partial z} \tag{5}$$

Differentiating (61.3) with respect to y and (61.4) with respect to x, and subtracting we have

$$2\frac{\partial^2 e_2}{\partial x\,\partial z} = \frac{\partial}{\partial y}\left[\frac{\partial e_4}{\partial x} - \frac{\partial e_5}{\partial y} + \frac{\partial e_6}{\partial z}\right] \tag{6}$$

Differentiating (61.4) with respect to z and (61.5) with respect to y, and subtracting we have

$$\frac{\partial^2 e_4}{\partial y\,\partial z} = \frac{\partial^2 e_2}{\partial z^2} + \frac{\partial^2 e_3}{\partial y^2} \tag{7}$$

Similarly from (61.3) and (61.5) we have

$$2\frac{\partial^2 e_3}{\partial y\,\partial x} = \frac{\partial}{\partial z}\left[\frac{\partial e_4}{\partial x} + \frac{\partial e_5}{\partial y} - \frac{\partial e_6}{\partial z}\right] \tag{8}$$

Similarly from the set of nine equations, by eliminating ω_x, ω_y, ω_z, we obtain the six identical relations between the components of strain which are

$$\left.\begin{aligned}
\frac{\partial^2 e_2}{\partial z^2} + \frac{\partial^2 e_3}{\partial y^2} &= \frac{\partial^2 e_4}{\partial y\,\partial z} \\
\frac{\partial^2 e_3}{\partial x^2} + \frac{\partial^2 e_1}{\partial z^2} &= \frac{\partial^2 e_5}{\partial x\,\partial z} \\
\frac{\partial^2 e_1}{\partial y^2} + \frac{\partial^2 e_2}{\partial x^2} &= \frac{\partial^2 e_6}{\partial x\,\partial y} \\
2\frac{\partial^2 e_1}{\partial y\,\partial z} &= \frac{\partial}{\partial x}\left[-\frac{\partial e_4}{\partial x} + \frac{\partial e_5}{\partial y} + \frac{\partial e_6}{\partial z}\right] \\
2\frac{\partial^2 e_2}{\partial z\,\partial x} &= \frac{\partial}{\partial y}\left[\frac{\partial e_4}{\partial x} - \frac{\partial e_5}{\partial y} + \frac{\partial e_6}{\partial z}\right] \\
2\frac{\partial^2 e_3}{\partial x\,\partial y} &= \frac{\partial}{\partial z}\left[\frac{\partial e_4}{\partial x} + \frac{\partial e_5}{\partial y} - \frac{\partial e_6}{\partial z}\right]
\end{aligned}\right\} \quad (9)$$

CHAPTER VI

Stress

62. Concept of stress: The material contained in the system considered will be subjected to forces, *e.g.* gravity. To balance these forces (and to overcome inertia if the material is in accelerated motion), forces must be exerted across the containing faces by the surrounding material. We need not concern ourselves here with the difficult physical problem of explaining the mechanism by which these forces are exerted; it is sufficient for our purposes to remark that the action, whatever it be, must be of a reciprocal nature, *i.e.* the force which is exerted on the contained material across a face S by the surrounding material must be equal to the force which is exerted by the contained material across the same face on the surrounding material. Similarly with the other faces. The concept of stress in general is simply that of balancing internal action and reaction between two parts of a body, the force which either part exerts on the other being one aspect of a stress.

The object of an analysis of stress is to determine the nature of the quantities by which stress at a point can be specified.

Consider now any plane area S in a system and containing a point O (x, y, z). Denote the normal to this plane (drawn in a certain sense) by ν. We assume that the force which is exerted across S is statically equivalent to a force P acting at O in a definite direction and a couple G about a definite axis. Now let the area of S approach zero in a continuous manner keeping the point O always within it. Then P and G will approach zero as a limit and the direction of the force towards a limiting direction (l, m, n).

Our concept is that

$$\lim_{\text{area } s \to 0} \frac{P}{S} = F \neq 0$$

$$\lim_{\text{area } s \to 0} \frac{G}{S} = 0$$

We define a vector quantity by the direction (l, m, n), the scalar value F, the position with respect to the body, *i.e.* whether acting on the positive or negative side of the body, and the dimension symbol m/lt^2 (force per unit area), and name it the *traction* across the plane ν at O. We denote the projections of this vector F on the x, y, z axes by X_ν, Y_ν, Z_ν respectively. The projection on the normal ν is

$$X_\nu \cos(x, \nu) + Y_\nu \cos(y, \nu) + Z_\nu \cos(z, \nu). \tag{1}$$

If this component traction, acting on the positive side of the body, is positive, *i.e.* its direction is away from the body, it is called a tension; if the component traction, acting on the positive side of the body, is negative, *i.e.* its direction is toward the body, it is called a pressure. If the component traction, acting on the negative side of the body, is negative it is a tension, if positive it is a pressure.

We shall assume that the nature of the action and reaction over the surfaces between two bodies in contact is the same as the nature of the action and reaction between two portions of the same body separated by an imagined surface.

Thus if we now allow O to move up to a point O^1, then X_ν, Y_ν, Z_ν tend to limiting values which we shall name the components of the *surface traction* at O^1. If we take a small neighborhood K about this point O^1 then X_ν K, Y_ν K, Z_ν K are the forces exerted across the element K of the bounding surface by some other body having contact with the body in question in the neighborhood of the point O^1.

The forces typified by the force of gravitation are in general proportional to the masses of the particles on which they act, and further their magnitude and direction are determined by the positions of these particles in the field of force. Let X, Y, Z be the components of the intensity of the field at any point O, m the mass of a particle that includes the point O, then mX, mY, mZ are the forces of the field that act on the particle.

The forces of the field may arise from the action of particles forming part of the body, as in the case of a body subjected to its own gravitation, or of particles outside the body, as in the case

of a body subject to the gravitational attraction of another body. In either case we call them *body forces.*

63. Interior and exterior forces. Six necessary conditions for equilibrium of a rigid body: The interior forces of a system are defined as the mutual actions of the different points of the system; we assume as a physical hypothesis that these actions are two by two equal and directly opposed after the principle of action and reaction.

Let (x_1, y_1, z_1), (x_2, y_2, z_2), $\cdots$, (x_n, y_n, z_n) be the coordinates of various points of the system and let $m_1, m_2, m_3, \cdots m_n$ be the masses of the neighborhoods

$$x_i + \delta > x_i > x_i - \delta$$
$$y_i + \eta > y_i > y_i - \eta$$
$$z_i + \epsilon > z_i > z_i - \epsilon$$
$$\text{where } i = 1, \cdots, n$$

respectively.

Consider an arbitrary neighborhood of mass m and coordinates $(x, y, z,)$. We can divide the forces applied at this neighborhood into two parts.

(1) Those which make up the interior forces acting on m; we shall call X_i, Y_i, Z_i the projections of one of these forces.

(2) Those which make up the exterior forces acting on the same m; we shall call X_e, Y_e, Z_e the projections of one of these forces.

Isolate one of these neighborhoods. If the system is in equilibrium, then each neighborhood or particle of the system is in equilibrium.

In order that such a particle be in equilibrium it is necessary and sufficient that the resultant of all the forces acting on the particle be zero. Projecting them on the coordinate axes we have then, for the equilibrium of the particle m, the three equations

$$\begin{aligned} \Sigma X_i + \Sigma X_e &= 0 \\ \Sigma Y_i + \Sigma Y_e &= 0 \\ \Sigma Z_i + \Sigma Z_e &= 0 \end{aligned} \qquad (1)$$

where Σ denotes the sum of the projections of all the interior forces applied at **m**.

Summing up for all particles of the system and letting the number of particles increase without limit, *i.e.* the sizes of the particles approach zero as a limit, we have

$$\begin{aligned} &\lim_{n \to \infty} \left[\sum_1^n \sum X_i + \sum_1^n \sum X_e \right] = 0 \\ &\lim_{n \to \infty} \left[\sum_1^n \sum Y_i + \sum_1^n \sum Y_e \right] = 0 \qquad (2) \\ &\lim_{n \to \infty} \left[\sum_1^n \sum Z_i + \sum_1^n \sum Z_e \right] = 0 \end{aligned}$$

and since the system is in equilibrium therefore

$$\lim_{n \to \infty} \sum_1^n \sum X_i = 0,$$

thus

$$\lim_{n \to \infty} \sum_1^n \sum X_e = 0;$$

similarly with the other two.

In equations (63.1) multiply the first by $-$ y, the second by x and add, this gives us

$$\Sigma\,(x\,Y_i - y\,X_i) + \Sigma\,(x\,Y_e - y\,X_e) = 0;$$

Summing up these equations for all the particles $\mathbf{m}_n$ and letting n increase without limit we have

$$\lim_{n \to \infty} \left[\sum_1^n \sum (x\,Y_i - y\,X_i) + \sum_1^n \sum (x\,Y_e - y\,X_e) \right] = 0.$$

Now

$$\lim_{n \to \infty} \sum^n \sum (x\,Y_i - y\,X_i)$$

represents the sum of the moments of all the interior forces with respect to the O z axis; this expression is then zero by hypothesis and therefore

$$\lim_{n \to \infty} \sum_{1}^{n} \sum (x\, Y_e - y\, X_e) = 0, \text{ etc.}$$

This gives us

$$\iiint_v X_i \, dv = 0, \quad \iiint_v Y_i \, dv = 0, \quad \iiint_v Z_i \, dv = 0,$$
$$\iiint_v (x\, Y_i - y\, X_i) \, dv = 0, \quad \iiint_v (y\, Z_i - z\, Y_i) \, dv = 0, \tag{3}$$
$$\iiint_v (z\, X_i - x\, Z_i) \, dv = 0,$$

or

$$\iiint_v X_e \, dv = 0, \quad \iiint_v Y_e \, dv = 0, \quad \iiint_v Z_e \, dv = 0,$$
$$\iiint_v (x\, Y_e - y\, X_e) \, dv = 0, \quad \iiint_v (y\, Z_e - z\, Y_e) \, dv = 0, \tag{4}$$
$$\iiint_v (z\, X_e - x\, Z_e) \, dv = 0.$$

i.e. For any system to be in equilibrium it is necessary that the sum of the projections of the interior or exterior forces on the three coordinate axes and the sum of their moments with respect to each of the three axes be zero.

64. Equations of equilibrium: The body forces applied to any portion of a system are statically equivalent to a single force applied at one point together with a couple. The components, parallel to the coordinate axes, of the force can be written

$$\iiint_v \rho\, X \, dv, \quad \iiint_v \rho\, Y \, dv, \quad \iiint_v \rho\, Z \, dv \tag{1}$$

where ρ, in general a function of x, y, z, is the density, and where the limits of integration are the bounding surfaces of the system.

The moments about the origin of these components will be

$$\iiint_V \rho\,[y\,X - x\,Y]\,dv,\quad \iiint_V \rho\,[z\,X - x\,Z]\,dv,\quad \iiint_V \rho\,[z\,Y - y\,Z]\,dv \tag{2}$$

Similarly the tractions applied at the surfaces, σ, of the system are equivalent to a resultant force and a couple.

The resultant force can be written

$$\iint X_\nu\,d\sigma,\quad \iint Y_\nu\,d\sigma,\quad \iint Z_\nu\,d\sigma \tag{3}$$

and the moments of these components about the origin

$$\iint [y\,X_\nu - x\,Y_\nu]\,d\sigma,\quad \iint [z\,X_\nu - x\,Z_\nu]\,d\sigma,\quad \iint [z\,Y_\nu - y\,Z_\nu]\,d\sigma. \tag{4}$$

Thus for equilibrium we have six equations, from (63.3),

$$\begin{aligned} \iiint \rho\,X\,dv + \iint X_\nu\,d\sigma &= 0 \\ \iiint \rho\,Y\,dv + \iint Y_\nu\,d\sigma &= 0 \\ \iiint \rho\,Z\,dv + \iint Z_\nu\,d\sigma &= 0 \end{aligned} \tag{5}$$

$$\begin{aligned} \iiint \rho\,[X\,y - x\,Y]\,dv + \iint [y\,X_\nu - x\,Y_\nu]\,d\sigma &= 0 \\ \iiint \rho\,[z\,X - x\,Z]\,dv + \iint [z\,X_\nu - x\,Z_\nu]\,d\sigma &= 0 \\ \iiint \rho\,[z\,Y - y\,Z]\,dv + \iint [z\,Y_\nu - y\,Z_\nu]\,d\sigma &= 0 \end{aligned} \tag{6}$$

65. Specification of stress at a point: Through any point O in a body there passes an ∞^2 system of planes and the complete specification of the stress at O involves the knowledge of the traction at O across all these planes.

We can express all these tractions in terms of the component tractions across planes parallel to the coordinate planes, and to obtain relations between these components. Let X_x, Y_x, Z_x denote the vector components of the traction across the plane x = constant, and a similar notation for the traction across the planes y = constant and z = constant.

The capital letter denotes the direction of the component traction and the subscript the plane across which it acts. The sense is such that X_x is positive when it is a tension, negative when it is a pressure.

Consider the equilibrium of a tetrahedral portion of the body, having one vertex at O (x, y, z), and the three edges that meet at this vertex parallel to the axis of coordinates. The remaining vertices are the intersections of these edges with a plane near to O. Denote the normal to this plane drawn away from O by ν, so that its direction cosines are $\cos(x,\nu)$, $\cos(y,\nu)$, $\cos(z,\nu)$. Denote the area of this plane by $\Delta\sigma$.

(1) The projections of the external forces acting on the volume Δv (body forces), are approximately

$$-\rho X\,\Delta v,\quad -\rho Y\,\Delta v,\quad -\rho Z\,\Delta v$$

where ρ denotes the density at O. [The approximation will become more accurate as the volume of the tetrahedron becomes less.]

(2) The projections of the stresses on the surfaces of the tetrahedron will be given approximately by

$$\begin{array}{lll}
-X_x\,\Delta\sigma\cos(x,\nu), & -Y_x\,\Delta\sigma\cos(x,\nu), & -Z_x\,\Delta\sigma\cos(x,\nu)\\
-X_y\,\Delta\sigma\cos(y,\nu), & -Y_y\,\Delta\sigma\cos(y,\nu), & -Z_y\,\Delta\sigma\cos(y,\nu)\\
-X_z\,\Delta\sigma\cos(z,\nu), & -Y_z\,\Delta\sigma\cos(z,\nu), & -Z_z\,\Delta\sigma\cos(z,\nu)\\
X_\nu\,\Delta\sigma & Y_\nu\,\Delta\sigma & Z_\nu\,\Delta\sigma
\end{array}$$

where $X_\nu\,\Delta\sigma$, $Y_\nu\,\Delta\sigma$, $Z_\nu\,\Delta\sigma$ are the resultant tractions across the face.

[This approximation will become more accurate as the volume of the tetrahedron becomes less.]

The sum of these projections on any axis must be zero since by hypothesis the body is in equilibrium; thus on O x we have

$$-\rho X \,\Delta \mathrm{v} - X_{\mathrm{x}}\,\Delta\sigma \cos(\mathrm{x}, \nu) - X_{\mathrm{y}}\,\Delta\sigma \cos(\mathrm{y}, \nu) - X_{\mathrm{z}}\,\Delta\sigma \cos(\mathrm{z}, \nu) + X_{\nu}\,\Delta\sigma = 0$$

Dividing by $\Delta\sigma$, and letting $\Delta\sigma$ approach zero as a limit, we have

$$X_{\nu} = X_{\mathrm{x}} \cos(\mathrm{x}, \nu) + X_{\mathrm{y}} \cos(\mathrm{y}, \nu) + X_{\mathrm{z}} \cos(\mathrm{z}, \nu) + \rho X \lim_{\Delta\sigma \to 0} \frac{\Delta \mathrm{v}}{\Delta\sigma}$$

But

$$\Delta \mathrm{v} = \frac{\Delta\sigma \cdot \mathrm{O}\nu}{4}$$

where $\mathrm{O}\,\nu$ is the height of the tetrahedron from the vertex O. Thus

$$\frac{\Delta \mathrm{v}}{\Delta\sigma} = \frac{\Delta\sigma}{\Delta\sigma}\left[\frac{\mathrm{O}\nu}{4}\right] = \frac{\mathrm{O}\nu}{4}$$

and therefore

$$\lim_{\Delta\sigma \to 0} \frac{\Delta \mathrm{v}}{\Delta\sigma} = \lim_{\Delta\sigma \to 0} \frac{\mathrm{O}\nu}{4} = 0.$$

Hence we have

$$X_{\nu} = X_{\mathrm{x}} \cos(\mathrm{x}, \nu) + X_{\mathrm{y}} \cos(\mathrm{y}, \nu) + X_{\mathrm{z}} \cos(\mathrm{z}, \nu) \qquad (1)$$

similarly

$$Y_{\nu} = Y_{\mathrm{x}} \cos(\mathrm{x}, \nu) + Y_{\mathrm{y}} \cos(\mathrm{y}, \nu) + Y_{\mathrm{z}} \cos(\mathrm{z}, \nu) \qquad (2)$$

$$Z_{\nu} = Z_{\mathrm{x}} \cos(\mathrm{x}, \nu) + Z_{\mathrm{y}} \cos(\mathrm{y}, \nu) + Z_{\mathrm{z}} \cos(\mathrm{z}, \nu) \qquad (3)$$

By these equations the traction across any plane through a point O is expressed in terms of the tractions across planes parallel to the coordinate planes.

On substituting the values of X_{ν}, Y_{ν}, Z_{ν} from equations (65.1, 2, 3) in equations (64.5), (64.6) we have the necessary

conditions for equilibrium with respect to the body forces and surface tractions given as

$$\left.\begin{aligned}
&\iiint \rho\,(y\,Z - z\,Y)\,dv + \iint \Big\{ \Big[\cos(x,\nu)\Big]\Big[y\,Z_x - z\,Y_x\Big] + \\
&\qquad \Big[\cos(y,\nu)\Big]\Big[y\,Z_y - z\,Y_y\Big] + \\
&\qquad \Big[\cos(z,\nu)\Big]\Big[y\,Z_z - z\,Y_z\Big]\Big\}\,d\sigma = 0 \\
&\iiint \rho\,(x\,Z - z\,X)\,dv + \iint \Big\{ \Big[\cos(x,\nu)\Big]\Big[x\,Z_x - z\,X_x\Big] + \\
&\qquad \Big[\cos(y,\nu)\Big]\Big[x\,Z_y - z\,X_y\Big] + \\
&\qquad \Big[\cos(z,\nu)\Big]\Big[x\,Z_z - z\,X_z\Big]\Big\}\,d\sigma = 0 \\
&\iiint \rho\,(x\,Y - y\,X)\,dv + \iint \Big\{ \Big[\cos(x,\nu)\Big]\Big[x\,Y_x - y\,X_x\Big] + \\
&\qquad \Big[\cos(y,\nu)\Big]\Big[x\,Y_y - y\,X_y\Big] + \\
&\qquad \Big[\cos(z,\nu)\Big]\Big[x\,Y_z - y\,X_z\Big]\Big\}\,d\sigma = 0.
\end{aligned}\right\} \quad (4)$$

$$\begin{aligned}
&\iiint \rho\,X\,dv + \iint \Big[X_x \cos(x,\nu) + X_y \cos(y,\nu) \\
&\qquad\qquad + X_z \cos(z,\nu)\Big]\,d\sigma = 0 \\
&\iiint \rho\,Y\,dv + \iint \Big[Y_x \cos(x,\nu) + Y_y \cos(y,\nu) \\
&\qquad\qquad + Y_z \cos(z,\nu)\Big]\,d\sigma = 0 \qquad (5) \\
&\iiint \rho\,Z\,dv + \iint \Big[Z_x \cos(x,\nu) + Z_y \cos(y,\nu) \\
&\qquad\qquad + Z_z \cos(z,\nu)\Big]\,d\sigma = 0
\end{aligned}$$

According to Green's Theorem we can write

$$\iint \left[X_x \cos(x, \nu) + X_y \cos(y, \nu) + X_z \cos(z, \nu) \right] d\sigma$$

$$= \iiint \left[\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} \right] dv \tag{6}$$

Equations (65.5) can thus be written

$$\iiint \left[\rho X + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} \right] dv = 0,$$

similarly for the other two.

Now this region v was wholly arbitrary. We can understand by v then any subregion of the original region v and this equation will still hold for this new subregion. Therefore the integral must vanish at every point of v, or

$$\left.\begin{aligned} &\rho X + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = 0 \\ &\text{Similarly} \\ &\rho Y + \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} = 0 \\ &\text{and} \\ &\rho Z + \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} = 0 \end{aligned}\right\} \tag{7}$$

Again, we have from Green's Theorem

$$\iint \Big[(y Z_x - z Y_x) \cos(x, \nu) + (y Z_y - z Y_y) \cos(y, \nu)$$
$$+ (y Z_z - z Y_z) \cos(z, \nu) \Big] \partial\sigma =$$

$$\iiint\left[\frac{\partial\,(y\,Z_x - z\,Y_x)}{\partial x} + \frac{\partial\,(y\,Z_y - z\,Y_y)}{\partial y} + \frac{\partial\,(y\,Z_z - z\,Y_z)}{\partial z}\right] dv$$

or simplifying,

$$= \iiint\left[y\left(\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z}\right) - z\left(\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z}\right)\right.$$
$$\left. + Z_y - Y_z\right] dv$$

Thus equations (65.4) can be written

$$\iiint\left[y\left(\rho Z + \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z}\right)\right.$$
$$\left. - z\left(\rho Y + \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z}\right) + Z_y - Y_z\right] dv = 0$$

and similarly for the other two.

But the coefficients of y and z are, from equations (65.7), identically zero, and the whole integrand is identically zero since the region **v** was wholly arbitrary, thus

$$Y_z - Z_y = 0$$

similarly (8)

$$Z_x = X_z, \text{ and } X_y = Y_x.$$

In order to simplify the writing of equations we shall use the following abbreviations:

$$X_1 = X_x$$
$$X_2 = Y_y$$
$$X_3 = Z_z$$
$$X_4 = Z_y = Y_z$$
$$X_5 = X_z = Z_x$$
$$X_6 = Y_x = X_y$$

which are then analogous to $e_1, \cdots, e_6$ which for very small deformations are given by the expressions

$$e_1 = \frac{\partial u}{\partial x}$$

$$e_2 = \frac{\partial v}{\partial y}$$

$$e_3 = \frac{\partial w}{\partial z}$$

$$e_4 = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$e_5 = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$e_6 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

CHAPTER VII

Thermodynamic treatment of systems homogeneously strained

In this chapter we shall limit ourselves to systems homogeneously strained and leave the treatment of systems in which the strain is not homogeneous to Chapter IX.

66. Definitions of work and heat: Now a homogeneously strained system has been defined as one the properties of which are the same at all points of the system, *i.e.* are constant with respect to the coordinates x, y, z.

For the systems treated previously we found that the work received by the systems could be defined by the integral $-\int p\,dv$. Now we discover that for stressed systems this expression will not define the work received by the systems. We must therefore extend the definition of work, such that it will express the work received by the stressed system and such that, when the stresses can be represented as pressures which are the same at all points and in all directions at any one point of the system, the expression will reduce to the integral $-\int p\,dv$.

Hence we define the work per unit mass, W, received by the system homogeneously strained or work of the path by the equation

$$W = \int_{s_0}^{s} \frac{1}{\rho} \sum_{i=1}^{6} X_i \frac{de_i}{ds}\, ds.$$

where $e_1, \cdots, e_6$ are functions of $t, X_1, \cdots, X_6$; $t, X_1 \cdots, X_6$ depend upon the path S. ρ is the density in the state of reference. Thus

$$W = \int_{s_0}^{s} \frac{1}{\rho} \left\{ \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial t} \frac{dt}{ds} + \sum_{k=1}^{6} \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial X_k} \frac{dX_k}{ds} \right\} ds$$

$$= \int_{t_0, X_{1_0}, \ldots, X_{6_0}}^{t, X_1, \ldots, X_6} \frac{1}{\rho} \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial t} dt + \frac{1}{\rho} \sum_{k=1}^{6} \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial X_k} d X_k$$

Now when $X_1 = X_2 = X_3 = -p$, $X_4 = X_5 = X_6 = 0$ we have

$$W = -\int_{s_0}^{s} \frac{1}{\rho} p \left[\frac{de_1}{ds} + \frac{de_2}{ds} + \frac{de_3}{ds} \right] ds$$

$$= -\int_{s_0}^{s} p \frac{dv}{ds} ds \text{ where } v = \frac{v}{m} = \frac{\text{unit volume}}{\text{density}}$$

Hence our definition satisfies the boundary conditions, namely that it will reduce to the previous definition of work received when the stresses reduce to pressures which are the same at all points and the same in all directions at any point of the system.

Similarly the heat per unit mass, Q, received by the system homogeneously strained or heat of the path is defined by the equation

$$Q = \int_{s_0}^{s} \frac{1}{\rho} \left\{ c_x \frac{dt}{ds} + \sum_{i=1}^{6} l_{x_i} \frac{dX_i}{ds} \right\} ds$$

$$= \int_{t_0, X_{1_0}, \ldots, X_{6_0}}^{t, X_1, \ldots, X_6} \frac{1}{\rho} c_x \, dt + \frac{1}{\rho} \sum l_{x_i} d X_i$$

where $c_x, l_{x_1}, \cdots, l_{x_6}$ are functions of $t, X_1, \cdots, X_6$ and $t, X_1, \cdots, X_6$ depend upon the path S. ρ is the density in the state of reference.

Now when $X_1 = X_2 = X_3 = -p$, $X_4 = X_5 = X_6 = 0$ this expression reduces to

$$Q = \int_{s_0}^{s} \left\{ c_p \frac{dt}{ds} + l_p \frac{dp}{ds} \right\} ds$$

since here

$$c_p = \frac{c_x}{\rho} \text{ and } l_p = \frac{1}{\rho} \left\{ l_{x_1} + l_{x_2} + l_{x_3} \right\}.$$

Hence our definition reduces to the previous definition of heat received when the stresses reduce to pressures which are the same at all points and the same in all directions at any point of the system.

67. Definition of $c_x, l_{x_1}, \cdots, l_{x_6}$: Along the path $t = \mathrm{S}$, $X_1 = \mathrm{K}_1$, $\cdots$, $X_6 = \mathrm{K}_6$ where $\mathrm{K}_1, \cdots, \mathrm{K}_6$ are constants,

$$\frac{dQ}{d\mathrm{S}} = \frac{dQ}{dt}$$

which is by definition the heat capacity per unit mass at constant stress. Thus $\dfrac{dQ}{dt} = \dfrac{1}{\rho} c_x$; and along this path

$$\frac{dW}{ds} = \frac{dW}{dt} = \frac{1}{\rho} \sum_{\mathrm{i}=1}^{6} X_{\mathrm{i}} \frac{\partial e_{\mathrm{i}}}{\partial t}.$$

Along the path $t = \mathrm{K}$, $X_1 = \mathrm{S}$, $X_2 = \mathrm{K}_2, \cdots, X_6 = \mathrm{K}_6$ where $\mathrm{K}, \mathrm{K}_2, \cdots, \mathrm{K}_6$ are constants,

$$\frac{dQ}{d\mathrm{S}} = \frac{dQ}{dX_1}$$

which is by definition the latent heat of change of stress per unit mass parallel to the x-axis, where temperature and the other stresses are constant. Thus $\dfrac{dQ}{dX_1} = \dfrac{1}{\rho} l_{x_1}$ and along this path

$$\frac{dW}{d\mathrm{S}} = \frac{dW}{dX_1} = \frac{1}{\rho} \sum_{\mathrm{i}=1}^{6} X_{\mathrm{i}} \frac{\partial e_{\mathrm{i}}}{\partial X_1}$$

Similarly for the other stresses.

68. Transformation of the heat and work integrals: Now by hypothesis X_{i}, $\mathrm{i} = 1, \cdots, 6$ can be expressed as functions of e_{i}, t,

$$X_{\mathrm{i}} = f_{\mathrm{i}}\,(e_1, \cdots, e_6, t),\ \mathrm{i} = 1, \cdots, 6.$$

Thus

$$\int_{t_0, X_{1_0}, \ldots, X_{6_0}}^{t, X_1, \ldots, X_6} \frac{1}{\rho} c_x \, dt + \frac{1}{\rho} \sum_{i=1}^{6} l_{x_i} \, dX_i$$

$$= \int_{t_0, e_{1_0}, \ldots, e_{6_0}}^{t, e_1, \ldots, e_6} \frac{1}{\rho} \left[c_x + \sum_{i=1}^{6} l_{x_i} \frac{\partial X_i}{\partial t} \right] dt + \frac{1}{\rho} \sum_{i=1}^{6} l_{x_i} \frac{\partial X_i}{\partial e_1} de_1 + \cdots$$

$$+ \frac{1}{\rho} \sum_{i=1}^{6} l_{x^i} \frac{\partial X_i}{\partial e_6} de_6$$

Let us define

$$c_x + \sum_{i=1}^{6} l_{x_i} \frac{\partial X_i}{\partial t} = c_e$$

$$\sum_{i=1}^{6} l_{x_i} \frac{\partial X_i}{\partial e_k} = l_{e_k}, \ k = 1, \cdots, 6$$

Then

$$\int_{t_0, X_{1_0}, \ldots, X_{6_0}}^{t, X_1, \ldots, X_6} \frac{1}{\rho} c_x \, dt + \frac{1}{\rho} \sum_{i=1}^{6} l_{x_i} \, dX_i = \int_{t_0, e_{1_0}, \ldots, e_{6_0}}^{t, e_1, \ldots, e_6} \frac{1}{\rho} c_e \, dt + \frac{1}{\rho} \sum_{i=1}^{6} l_{e_i} \, de_i$$

The transformation of the work integral, as we have seen, gives us

$$\int_{t_0, X_{1_0}, \ldots, X_{6_0}}^{t, X_1, \ldots, X_6} \frac{1}{\rho} \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial t} dt + \frac{1}{\rho} \sum_{k=1}^{6} \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial X_k} dX_k = \int_{t_0, e_{1_0}, \ldots, e_{6_0}}^{t, e_1, \ldots, e_6} \frac{1}{\rho} \sum_{i=1}^{6} X_i \, de_i$$

Along the path $t = S$, $e_i = K_i^1$, $i = 1, \cdots, 6$ where K_i^1 are constants, $\frac{dQ}{dS} = \frac{dQ}{dt}$ which is by definition the heat capacity per unit mass at constant strain, that is, where e_i, $i = 1, \cdots, 6$ are constants. Thus $\frac{dQ}{dt} = \frac{1}{\rho} c_e$. And along this path $\frac{dW}{dS} = \frac{dW}{dt} = 0$. Along the

path $t = K$, $e_1 = S$, $e_2 = K_2{}^1, \cdots, e_6 = K_6{}^1$, $\frac{dQ}{dS} = \frac{dQ}{de_1}$ which is by definition the latent heat of change of strain along the x-axis per unit mass where temperature and the other strains are constants. Thus $\frac{dQ}{de_1} = \frac{1}{\rho} l_{e_1}$. And along this path $\frac{dW}{dS} = \frac{dW}{de_1} = \frac{1}{\rho} X_1$. Similarly for the other strains.

69. The first law of thermodynamics: The first law of thermodynamics, for a unit volume system *homogeneously strained*, is expressed by the equation

$$\rho\epsilon(t, X_1, \cdots, X_6) - \rho_0\epsilon(t_0, X_{1_0}, \cdots, X_{6_0}) =$$

$$\int_{t_0, X_{1_0}, \ldots, X_{6_0}}^{t, X_1, \ldots, X_6} \left[c_x + \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial t}\right] dt + \sum_{k=1}^{6} \left[l_{xk} + \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial X_k}\right] dX_k \tag{2}$$

We complete the definition of $\rho\epsilon(t, X_1, \cdots, X_6)$ by defining the energy per unit volume at $t_0, X_{1_0}, \cdots, X_{6_0}$ as zero,

$$\rho_0\epsilon(t_0, X_{1_0}, \cdots, X_{6_0}) = 0, \tag{3}$$

where ρ_0 denotes the density in the state of reference.

Hence

$$\rho\epsilon_t(t, X_1, \cdots, X_6) = c_x + \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial t} \tag{4}$$

$$\rho\epsilon_{xk}(t, X_1, \cdots, X_6) = l_{xk} + \sum X_i \frac{\partial e_i}{\partial X_k}, \quad k = 1, \cdots, 6 \tag{5}$$

And from the second derivatives of $\rho\epsilon$ we have

$$\frac{\partial}{\partial X_1}\left[c_x + \sum X_i \frac{\partial e_i}{\partial t}\right] = \frac{\partial}{\partial t}\left[l_{x_1} + \sum X_i \frac{\partial e_i}{\partial X_1}\right]$$

or

$$\left(\frac{\partial c_x}{\partial X_1}\right)_{t, X_2, \ldots, X_6} = \left(\frac{\partial l_{x_1}}{\partial t}\right)_{X_1, \ldots, X_6} - \left(\frac{\partial e_1}{\partial t}\right)_{X_1, \ldots, X_6}$$

Similarly

$$\frac{\partial c_x}{\partial X_n} = \frac{\partial l_{x_n}}{\partial t} - \frac{\partial e_n}{\partial t}, \ n = 1, \cdots, 6. \tag{6}$$

$$\frac{\partial}{\partial X_2}\left[l_{x_1} + \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial X_1}\right] = \frac{\partial}{\partial X_1}\left[l_{x_2} + \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial X_2}\right]$$

or

$$\left(\frac{\partial l_{x_1}}{\partial X_2}\right)_{t,\, X_1,\, X_3,\, \ldots,\, X_6} - \left(\frac{\partial l_{x_2}}{\partial X_1}\right)_{t,\, X_2,\, \ldots,\, X_6} =$$

$$\left(\frac{\partial e_1}{\partial X_2}\right)_{t,\, X_1,\, X_3,\, \ldots,\, X_6} - \left(\frac{\partial e_2}{\partial X_1}\right)_{t,\, X_2,\, \ldots,\, X_6}$$

Similarly

$$\frac{\partial l_{x_m}}{\partial X_n} - \frac{\partial l_{x_n}}{\partial X_m} = \frac{\partial e_m}{\partial X_n} - \frac{\partial e_n}{\partial X_m}, \ (m, n) = 1, \cdots, 6 \tag{7}$$

70. Transformation of the energy integral: By hypothesis $X_i = f_i(e_1, \cdots, e_6, t)$, $i = 1, \cdots, 6$.

Thus

$$\int_{t_0,\, X_{1_0},\, \ldots,\, X_{6_0}}^{t,\, X_1,\, \ldots,\, X_6} \left[c_x + \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial t}\right] dt + \sum_{k=1}^{6}\left[l_{x_k} + \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial X_k}\right] d X_k$$

$$= \int_{t_0,\, e_{1_0},\, \ldots,\, e_{6_0}}^{t,\, e_1,\, \ldots,\, e_6} \left\{c_x + \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial t} + \sum_{k=1}^{6}\left[l_{x_k} + \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial X_k}\right]\frac{\partial X_k}{\partial t}\right\} dt$$

$$+ \sum_{k=1}^{6}\left[l_{x_k} + \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial X_k}\right]\frac{\partial X_k}{\partial e_1} de_1$$

$$+ \cdots + \sum_{k=1}^{6}\left[l_{x_k} + \sum X_i \frac{\partial e_i}{\partial X_k}\right]\frac{\partial X_k}{\partial e_6} de_6$$

$$= \int_{t_0,\, e_{1_0},\, \ldots,\, e_{6_0}}^{t,\, e_1,\, \ldots,\, e_6} c_e \, dt + \sum_{i=1}^{6}\left[l_{e_i} + X_i\right] de_i \tag{1}$$

since, by definition

$$c_x + \sum_{k=1}^{6} l_{xk} \frac{\partial X_k}{\partial t} = c_e$$

$$\sum_{k=1}^{6} l_{xk} \frac{\partial X_k}{\partial e_i} = l_{e_i}, \ i = 1, \cdots, 6.$$

Hence

$$\rho \, \epsilon_t \, (t, e_1, \cdots, e_6) = c_e \tag{2}$$

$$\rho \, \epsilon_{e_i} \, (t, e_1, \cdots, e_6) = l_{e_i} + X_i, \ i = 1, \cdots, 6. \tag{3}$$

And from the second derivatives of ϵ we have

$$\left(\frac{\partial c_e}{\partial e_1}\right)_{t,\, e_2, \, \dots, \, e_6} = \left(\frac{\partial l_{e_1}}{\partial t}\right)_{e_1, \, \dots, \, e_6} + \left(\frac{\partial X_1}{\partial t}\right)_{e_1, \, \dots, \, e_6}$$

or

$$\frac{\partial c_e}{\partial e_n} = \frac{\partial l_{e_n}}{\partial t} + \frac{\partial X_n}{\partial t}, \ n = 1, \cdots, 6. \tag{4}$$

Similarly

$$\frac{\partial l_{e_m}}{\partial e_n} - \frac{\partial l_{e_n}}{\partial e_m} = \frac{\partial X_n}{\partial e_m} - \frac{\partial X_m}{\partial e_n}, \ (m, n) = 1, \cdots, 6. \tag{5}$$

71. The second law of thermodynamics: The second law of thermodynamics, for the unit volume system *homogeneously strained,* is expressed by the equation

$$\rho \, \eta \, (t, X_1, \cdots, X_6) - \rho_0 \, \eta \, (t_0, X_{1_0}, \cdots, X_{6_0})$$
$$= \int_{t_0, X_{1_0}, \dots, X_{6_0}}^{t, X_1, \dots, X_6} \frac{c_x}{\theta} \, dt + \sum_{i=1}^{6} \frac{l_{x_i}}{\theta} \, d X_i, \text{ where } \theta \neq 0, \tag{2}$$

and where ρ_0 denotes the density in the state of reference.

We assume, as a physical hypothesis, that for systems in stable equilibrium

$$\lim_{\theta \to 0} \frac{c_x}{\theta} = 0 \tag{3}$$

We thus extend the definition of the entropy function by defining the entropy at $\theta = \Gamma(t_0) = 0, X_{1_0} \cdots, X_{6_0}$, as zero, *i.e.*

$$\rho_0 \eta (t_0, X_{1_0}, \cdots, X_{6_0}) = 0. \tag{4}$$

Now

$$\rho \eta_t (t, X_1, \cdots, X_6) = \frac{c_x}{\theta} \tag{5}$$

$$\rho \eta_{x_i} (t, X_1, \cdots, X_6) = \frac{l_{x_i}}{\theta}, \ i = 1, \cdots, 6. \tag{6}$$

And from the second derivatives of $\rho \eta$ we have

$$\frac{\partial c_x}{\partial X_k} = \frac{\partial l_{x_k}}{\partial t} - \frac{l_{x_k}}{\theta} \tag{7}$$

and

$$\frac{\partial l_{x_n}}{\partial X_m} - \frac{\partial l_{x_m}}{\partial X_n} = 0, \ (m, n) = 1, \cdots, 6. \tag{8}$$

Thus from (69.6) and (71.7)

$$l_{x_k} = \theta \left(\frac{\partial e_k}{\partial \theta} \right)_{X_1, \ldots, X_6}$$

and from (69.7) and (71.8)

$$\frac{\partial e_n}{\partial X_m} = \frac{\partial e_m}{\partial X_n}$$

72. Transformation of the entropy integral: By hypothesis $X_i = f_i (e_1, \cdots, e_6, t)$, $i = 1, \cdots, 6$.

Thus

$$\int_{t_0, X_{1_0}, \ldots, X_{6_0}}^{t, X_1, \ldots, X_6} \frac{c_x}{\theta} dt + \sum_{i=1}^{6} \frac{l_{x_i}}{\theta} d X_i = \int_{t_0, e_{1_0}, \ldots, e_{6_0}}^{t, e_1, \ldots, e_6} \left[\frac{c_x}{\theta} + \sum_{i=1}^{6} \frac{l_{x_i}}{\theta} \frac{\partial X_i}{\partial t} \right] dt +$$

$$\sum_{k=1}^{6} \sum_{i=1}^{6} \frac{l_{x_i}}{\theta} \frac{\partial X_i}{\partial e_k} de_k$$

$$= \int_{t_0, e_{1_0}, \ldots, e_{6_0}}^{t, e_1, \ldots, e_6} \frac{c_e}{\theta}\, dt + \sum_{i=1}^{6} \frac{l_{e_i}}{\theta}\, de_i \tag{1}$$

since by definition

$$c_x + \sum_{i=1}^{6} l_{x_i} \frac{\partial X_i}{dt} = c_e$$

$$\sum_{i=1}^{6} l_{x_i} \frac{\partial X_i}{\partial e_k} = l_{e_k}, \; k = 1, \cdots, 6.$$

Hence

$$\rho\, \eta_t\, (t, e_1, \cdots, e_6) = \frac{c_e}{\theta} \tag{2}$$

$$\rho\, \eta_{e_n}\, (t, e_1, \cdots, e_6) = \frac{l_{e_n}}{\theta} \tag{3}$$

And from the second derivatives of $\rho\, \eta$ we have

$$\frac{\partial c_e}{\partial e_n} = \frac{\partial l_{e_n}}{\partial \theta} - \frac{l_{e_n}}{\theta}, \; n = 1, \cdots, 6 \tag{4}$$

and

$$\frac{\partial l_{e_n}}{\partial e_m} = \frac{\partial l_{e_m}}{\partial e_n} \tag{5}$$

Thus from (70.4) and (72.4) we have

$$l_{e_n} = -\,\theta \left(\frac{\partial X_n}{\partial \theta} \right)_{e_1, \ldots, e_6}, \; n = 1, \cdots, 6.$$

And from (70.5) and (72.5) we have

$$\frac{\partial X_n}{\partial e_m} = \frac{\partial X_m}{\partial e_n}$$

73. Derivation of Gibbs' equation 12 for strained systems: Assuming that we can solve for θ as a single valued continuous

function of $\eta, e_1, \cdots, e_6$ and obtain ϵ as a continuous function of $\eta, e_1, \cdots, e_6$ we obtain

$$\epsilon_\eta (\eta, e_1, \cdots, e_6) = \left(\frac{\partial \epsilon}{\partial \theta}\right)_{e_1, \ldots, e_6} \left(\frac{\partial \theta}{\partial \eta}\right)_{e_1, \ldots, e_6}$$

$$= \frac{1}{\rho} c_e \cdot \rho \frac{\theta}{c_e} = \theta$$

$$\epsilon_{e_i} (\eta, e_1, \cdots, e_6) = \frac{\partial \epsilon}{\partial e_i} + \frac{\partial \epsilon}{\partial \theta} \frac{\partial \theta}{\partial e_i}$$

$$= \frac{1}{\rho} l_{e_i} + \frac{1}{\rho} X_i + \frac{1}{\rho} c_e \left(- \frac{l_{e_i}}{\theta} \cdot \frac{\theta}{c_e} \right)$$

$$= \frac{1}{\rho} X_i, \ i = 1, \cdots, 6.$$

Thus

$$d\epsilon = \theta \, d\eta + \frac{1}{\rho} \sum_{i=1}^{6} X_i \, de_i$$

or

$$d\boldsymbol{\varepsilon} = \theta \, d\mathbf{n} + \mathbf{v} \sum_{i=1}^{6} X_i \, de_i$$

where $\mathbf{v}$ denotes the total volume in the state of reference.

74. Differential and partial derivatives of Gibbs' Zeta: By definition

$$\zeta = \epsilon - \theta \eta - \frac{1}{\rho} \sum_{i=1}^{6} X_i \, e_i$$

Hence

$$\rho \zeta = \rho \zeta (\theta, X_1, \cdots, X_6)$$

Thus

$$\rho \zeta_\theta (\theta, X_1, \cdots, X_6) = \rho \left(\frac{\partial \epsilon}{\partial \theta}\right)_{X_1, \ldots, X_6} - \rho \eta - \rho \theta \left(\frac{\partial \eta}{\partial \theta}\right)_{X_1, \ldots, X_6}$$

$$- \sum_{i=1}^{6} X_i \left(\frac{\partial e_i}{\partial \theta}\right)_{X_1, \ldots, X_6}$$

$$= c_x + \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial \theta} - \rho \eta - c_x - \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial \theta}$$

$$= - \rho \eta$$

$$\rho \zeta_{x_n}(\theta, X_1, \cdots, X_6) = \rho \frac{\partial \epsilon}{\partial X_n} - \rho \theta \frac{\partial \eta}{\partial X_n} - e_n - \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial X_n}$$

$$= l_{x_n} + \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial X_n} - l_{x_n} - e_n - \sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial X_n}$$

$$= - e_n, \ n = 1, \cdots, 6.$$

Hence

$$d\zeta = - \eta \, d\theta - \frac{1}{\rho} \sum_{i=1}^{6} e_i \, d X_i$$

and

$$\frac{\partial \eta}{\partial X_n} = \frac{1}{\rho} \frac{\partial e_n}{\partial \theta}, \ n = 1, \cdots, 6.$$

Thus

$$d\boldsymbol{\zeta} = - \mathbf{n} \, d\theta - \mathbf{v} \sum_{i=1}^{6} e_i \, d X_i$$

where $\mathbf{v}$ denotes the total volume in the state of reference.

75. Differential and partial derivatives of enthalpy or Gibbs' Chi: By definition

$$\chi = \epsilon - \frac{1}{\rho} \sum_{i=1}^{6} X_i e_i$$

$$= \chi(\eta, X_1, \cdots, X_6)$$

Thus

$$\chi_\eta(\eta, X_1, \cdots, X_6) = \left(\frac{\partial \epsilon}{\partial \eta}\right)_{e_1, \ldots, e_6} + \frac{1}{\rho}\sum_{i=1}^{6} \frac{\partial \epsilon}{\partial e_i}\frac{\partial e_i}{\partial \eta} - \frac{1}{\rho}\sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial \eta}$$

$$= \theta + \frac{1}{\rho}\sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial \eta} - \frac{1}{\rho}\sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial \eta}$$

$$= \theta$$

$$\chi_{x_n}(\eta, X_1, \cdots, X_6) = \sum_{i=1}^{6} \frac{\partial \epsilon}{\partial e_i}\frac{\partial e_i}{\partial X_n} - \frac{1}{\rho} e_n - \frac{1}{\rho}\sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial X_n}$$

$$= \frac{1}{\rho}\sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial X_n} - \frac{1}{\rho} e_n - \frac{1}{\rho}\sum_{i=1}^{6} X_i \frac{\partial e_i}{\partial X_n}$$

$$= -\frac{1}{\rho} e_n.$$

Thus

$$d\chi = \theta\, d\eta - \frac{1}{\rho}\sum_{i=1}^{6} e_i\, d X_i$$

and

$$\frac{\partial \theta}{\partial X_n} = -\frac{1}{\rho}\frac{\partial e_n}{\partial \eta}$$

Hence

$$d\boldsymbol{\chi} = \theta\, d\mathbf{n} - \mathbf{v}\sum_{i=1}^{6} e_i\, d X_i$$

where $\mathbf{v}$ denotes the total volume in the state of reference.

76. Differential and partial derivatives of Gibbs' Psi: By definition

$$\psi = \epsilon - \theta\eta$$

Hence

$$\rho\psi = \rho\psi(\theta, e_1, \cdots, e_6)$$

Thus

$$\rho\,\psi_\theta\,(\theta, e_1, \cdots, e_6) = \rho\left(\frac{\partial\epsilon}{\partial\theta}\right)_{e_1, \ldots, e_6} - \rho\,\eta - \rho\,\theta\left(\frac{\partial\eta}{\partial\theta}\right)_{e_1, \ldots, e_6}$$

$$= C_e - \rho\,\eta - C_e$$

$$= -\rho\,\eta$$

$$\rho\,\psi_{e_n}\,(\theta, e_1, \cdots, e_6) = \rho\left(\frac{\partial\epsilon}{\partial e_n}\right) - \rho\,\theta\left(\frac{\partial\eta}{\partial e_n}\right)$$

$$= l_{e_n} + X_n - l_{e_n}$$

$$= X_n, \mathrm{n} = 1, \cdots, 6.$$

Hence

$$d\,\psi = -\,\eta\,d\theta + \frac{1}{\rho}\sum_{i=1}^{6} X_i\,de_i$$

and

$$-\frac{\partial\eta}{\partial e_n} = \frac{1}{\rho}\frac{\partial X_n}{\partial\theta}, \mathrm{n} = 1, \cdots, 6.$$

Thus

$$d\boldsymbol{\psi} = -\,\mathbf{n}\,d\theta + \mathbf{v}\sum_{i=1}^{6} X_i\,de_i$$

where $\mathbf{v}$ denotes the total volume in the state of reference.

CHAPTER VIII

The stress-strain relations for isothermal changes of state

77. Generalized Hooke's Law of the proportionality of stress and strain: We shall now consider the special case where the changes of state take place at constant temperature.

Then

$$X_i = f_i\,(e_1, e_2, e_3, e_4, e_5, e_6),\ i = 1, \cdots, 6, \tag{1}$$

$X_1, \cdots, X_6$ being zero in the state of reference, *i.e.* when $e_1, \cdots, e_6$ are zero. Developing these according to the McLaurin expansion we obtain series proceeding according to increasing positive powers of $e_1, \cdots, e_6$. If we neglect terms containing products and powers of $e_1, \cdots, e_6$ we shall have for $X_1, \cdots, X_6$ linear homogeneous expressions in $e_1, \cdots, e_6$.

In the following discussion we shall assume as a physical hypothesis that the GENERALIZED HOOKE'S LAW OF THE PROPORTIONALITY OF STRESS AND STRAIN holds, *i.e.* that each of the six components of stress at any point of a body is a linear function of the six components of strain at the point.

78. Strain-energy function, $\frac{W}{V}$, for isothermal changes of state: From the second law we know, for any closed path consisting of a continuous series of equilibrium states in which the temperature remains constant, the heat of this path must be zero.

$$\mathbf{Q}_S = \int_{\eta_0}^{\eta_1} \theta_1\, d\mathbf{n} = \theta_1\,(\mathbf{n}_1 - \mathbf{n}_0)$$

where $\mathbf{Q}_S$ is the heat of the path S from the state in which $\mathbf{n} = \mathbf{n}_0$ to the state in which $\mathbf{n} = \mathbf{n}_1$.

$$\mathbf{Q}_\sigma = \int_{\eta_1}^{\eta_0} \theta_1\, d\mathbf{n} = \theta_1\,(\mathbf{n}_0 - \mathbf{n}_1)$$

where Q_σ is the heat of the path σ from the state in which $\mathfrak{n} = \mathfrak{n}_1$ to the state in which $\mathfrak{n} = \mathfrak{n}_0$.

Thus

$$Q_S + Q_\sigma = \theta_1 [(\mathfrak{n}_1 - \mathfrak{n}_0) + (\mathfrak{n}_0 - \mathfrak{n}_1)] = 0.$$

Hence the heat of the path for a continuous series of *isothermal equilibrium states* is independent of the path and thus, for a fixed initial state, is a function of the final state only.

Therefore this must also be true for the work of the path and so for an isothermal change of state we have, for the work *received by the system*,

$$\frac{W}{V}(e_1, \cdots, e_6) = \int_{e_{1_0}, \ldots, e_{6_0}}^{e_1, \ldots, e_6} \sum_{i=1}^{6} X_i \, de_i$$

the line integral being extended along any path connecting the points $(e_{1_0}, \cdots, e_{6_0})$ and $(e_1, \cdots, e_6)$.

79. Strain-energy function, $\frac{W}{V}$, for adiabatic changes of state: Now for changes of state that take place adiabatically, *i.e.* where no heat is gained or lost by any part of the system, we have for the work $\frac{W}{V}$ received by the system of unit volume,

$$\frac{W}{V} = \int_{t_0, e_{1_0}, \ldots, e_{6_0}}^{t_1, e_1, \ldots, e_6} \sum_{i=1}^{6} X_i \, de_i$$

which, from the first law of thermodynamics, is independent of the path for, in this case.

$$\frac{\varepsilon}{V}(t_1, e_1, \cdots, e_6) - \frac{\varepsilon}{V}(t_0, e_{1_0}, \cdots, e_{6_0}) = \frac{W}{V}$$

Now since we have assumed the stresses to be linear functions of the strains, and since $\frac{\partial}{\partial e_i}\left(\frac{W}{V}\right) = X_i$, $i = 1, \cdots, 6$, the strain-energy function $\frac{W}{V}$ is therefore a homogeneous quadratic function of the strains.

80. Static *vs.* dynamic methods of determining the stress-strain relations. Relation between W for isothermal and W for adiabatic changes of state. Now X_i are in general functions of $t, e_1, \cdots, e_6$. Therefore the strain-energy function $\frac{W}{v}$ will not as a rule be the same for the adiabatic and the isothermal changes of state. Thus from a theoretical point of view X_i, $(i = 1, \cdots, 6)$, as determined experimentally by statical methods (involving isothermal changes of state) will differ from X_i, $i = 1, \cdots, 6$, as determined experimentally by dynamical methods (involving adiabatic changes of state). This has experimentally been shown to be the case although the measured differences were not very large.

Now stress-strain relations, as determined by statical methods, have been applied, in some cases it seems rather indiscriminately, to adiabatic changes of state. Therefore it might be well to stress the point that, when using these two strain-energy functions as interchangeable, the burden of proving that the discrepancies between the two are negligible for the problem under consideration rests with the user.

81. The elastic coefficients or "elastic constants" of the system: From §77, we have X_i, $i = 1, \cdots, 6$ given as linear homogeneous expressions in e_h, $h = 1, \cdots, 6$. Hence we can write

$$X_i = \sum_{h=1}^{6} c_{ih}\, e_h,\ i = 1, \cdots, 6. \tag{1}$$

In these equations the coefficients c_{ih} of e_h number 36. These coefficients depend on the constitution of the body at the point P (x, y, z) in question; for a body whose constitution varies from point to point these coefficients will be functions of x, y, z.

Assume that the system is homogeneous in the reference state, *i.e.* its constitution is the same at each point. The density ρ, and the 36 coefficients are then constant for the system under observation.

If we substitute the values of $X_1, \cdots, X_6$ from (81.1) in

$$d\,\frac{W}{v} = \sum_{i=1}^{6} X_i\, de_i$$

we find that the "Elastic Constants" c_{ih} are the coefficients of a homogeneous quadratic function $\frac{2W}{v}$ where $\frac{W}{v}$ is the strain-energy function; they are therefore connected by relations which insure the existence of the function. These relations are of the form

$$c_{ih} = c_{hi}, \ (h, i) = 1, \cdots, 6 \tag{2}$$

and the number of constants is reduced from 36 to 21.

$$\frac{2W}{v} = \sum_{i=1}^{6} \sum_{h=1}^{6} c_{ih}\, e_i\, e_h \text{ where } c_{ih} = c_{hi}. \tag{3}$$

82. The stress-strain relations for isotropic bodies: In isotropic solids every plane is a plane of symmetry and every axis is an axis of symmetry, and the corresponding rotation may be of any amount. Hence the equations connecting stress components are independent of the axes of coordinates, *i.e.* of direction. Thus $\frac{W}{v}$ is invariant for all transformations from one set of orthogonal axes to another.

We shall assume the theorem[1] that in the transformation of the quadratic expression the following are the only invariants with respect to transformations from one set of rectangular axes to another

$$\begin{aligned} &e_1 + e_2 + e_3 \\ &e_2\, e_3 + e_3\, e_1 + e_1\, e_2 - \frac{1}{4}(e_4^2 + e_5^2 + e_6^2) \\ &e_1\, e_2\, e_3 + \frac{1}{4}(e_4\, e_5\, e_6 - e_1\, e_4^2 - e_2\, e_5^2 + e_3\, e_6^2) \end{aligned} \tag{1}$$

In making the transformation for isotropic solids we thus find only two invariants of the strain of first or second degrees and thus the strain-energy-function $\frac{W}{v}$ is

[1] G. Salmon, Geometry of Three Dimensions, 1882, 4th ed., p. 66.

$$\frac{W}{v} = \frac{1}{2} c_{11} (e_1^2 + e_2^2 + e_3^2) + c_{12} (e_2 e_3 + e_3 e_1 + e_1 e_2) +$$

$$\frac{1}{4} (c_{11} - c_{12}) (e_4^2 + e_5^2 + e_6^2)$$

$$= \frac{1}{2} c_{11} (e_1 + e_2 + e_3)^2 - \frac{1}{2} c_{11} (2 e_1 e_2 + 2 e_1 e_3 + 2 e_2 e_3) +$$

$$c_{12} (e_2 e_3 + e_3 e_1 + e_1 e_2) + \frac{1}{4} (c_{11} - c_{12}) (e_4^2 + e_5^2 + e_6^2)$$

$$= \frac{1}{2} c_{11} (e_1 + e_2 + e_3)^2 - (c_{11} - c_{12}) (e_1 e_2 + e_2 e_3 + e_3 e_1) +$$

$$\frac{1}{4} (c_{11} - c_{12}) (e_4^2 + e_5^2 + e_6^2) \qquad (2)$$

Let $c_{11} = c_{12} + 2 R$ and $\Delta = e_1 + e_2 + e_3$.
Then

$$\frac{2 W}{v} = (c_{12} + 2 R) \Delta^2 + R (e_4^2 + e_5^2 + e_6^2 - 4 e_1 e_2 - 4 e_2 e_3 - 4 e_3 e_1) \qquad (3)$$

The stress components for an isotropic body can then be written in the form

$$\begin{aligned} X_1 (= X_x) &= c_{11} e_1 + c_{12} e_2 + c_{13} e_3 + c_{14} e_4 + c_{15} e_5 + c_{16} e_6 \\ &= c_{11} e_1 + c_{12} e_2 + c_{12} e_3 \\ &= (c_{12} + 2 R) e_1 + c_{12} (e_2 + e_3) \\ &= 2 R e_1 + c_{12} \Delta \qquad (4) \end{aligned}$$

$$\begin{aligned} X_2 (= X_y) &= c_{21} e_1 + c_{22} e_2 + c_{23} e_3 + c_{24} e_4 + c_{25} e_5 + c_{26} e_6 \\ &= c_{12} e_1 + c_{11} e_2 + c_{12} e_3 \\ &= 2 R e_2 + c_{12} \Delta \qquad (5) \end{aligned}$$

$$\begin{aligned} X_3 (= X_z) &= c_{31} e_1 + c_{32} e_2 + c_{33} e_3 + c_{34} e_4 + c_{35} e_5 + c_{36} e_6 \\ &= c_{12} e_1 + c_{12} e_2 + c_{11} e_3 \\ &= 2 R e_3 + c_{12} \Delta \qquad (6) \end{aligned}$$

$$
\begin{aligned}
X_4\,(=Z_y=Y_z) &= c_{41}\,e_1 + c_{42}\,e_2 + c_{43}\,e_3 + c_{44}\,e_4 + c_{45}\,e_5 + c_{46}\,e_6 \\
&= c_{44}\,e_4 \\
&= \frac{1}{2}\,(c_{11}-c_{12})\,e_4 \\
&= R\,e_4 \qquad (7)
\end{aligned}
$$

Similarly

$$X_5\,(=X_z=Z_x) = R\,e_5 \qquad (8)$$

$$X_6\,(=Y_x=X_y) = R\,e_6 \qquad (9)$$

83. Moduli of elasticity for isotropic substances. Let

$$X_1 = X_2 = X_3 = -P$$

$$X_4 = X_5 = X_6 = 0$$

From equations (82.4), (82.5), (82.6) we have

$$-P = c_{12}\,\Delta + 2\,R\,e_1 \qquad (1)$$

$$-P = c_{12}\,\Delta + 2\,R\,e_2 \qquad (2)$$

$$-P = c_{12}\,\Delta + 2\,R\,e_3 \qquad (3)$$

Subtracting (83.1) from (83.2), etc. we have

$$e_1 = e_2 = e_3$$

Adding (83.1), (83.2), (83.3) we have

$$
\begin{aligned}
-3\,P &= 3\,c_{12}\,\Delta + 2\,R\,\Delta = 3\,c_{12}\,(e_1+e_2+e_3) + 2\,R\,(e_1+e_2+e_3) \\
&= 9\,c_{12}\,e_1 + 6\,R\,e_1
\end{aligned}
$$

Thus

$$e_1 = e_2 = e_3 = \frac{-P}{3\,c_{12} + 2\,R} \qquad (4)$$

The cubical compression is then

$$-\Delta = \frac{3\,P}{3\left(c_{12} + \frac{2}{3}R\right)} = \frac{P}{c_{12} + \frac{2}{3}R} \qquad (5)$$

We define a quantity $K = -\frac{P}{\Delta}$ as the *bulk modulus* or *modulus of compression.*

Thus

$$K = c_{12} + \frac{2}{3} R.$$

The *compressibility*, β, is defined as $\frac{1}{K}$. (6)

Now let $X_1 = T$, where T = tensional stress, and $X_2 = X_3 = X_4 = X_5 = X_6 = 0$

From (82.4, 5, 6) we have

$$T = c_{12}(e_1 + e_2 + e_3) + 2\,R\,e_1 \tag{7}$$

$$0 = c_{12}(e_1 + e_2 + e_3) + 2\,R\,e_2 \tag{8}$$

$$0 = c_{12}(e_1 + e_2 + e_3) + 2\,R\,e_3 \tag{9}$$

From (83.8, 9) we have $e_2 = e_3$; thus from (83.8) or (83.9) we have, with $e_2 = e_3$

$$e_2 = e_3 = \frac{-\,e_1\,c_{12}}{2\,(c_{12} + R)}. \tag{10}$$

Substituting this value in (83.7) we have

$$\begin{aligned} T &= e_1(2\,R + c_{12}) - \frac{c_{12}^2\,e_1}{c_{12} + R} \\ &= \frac{e_1\,3\,R\,c_{12} + e_1\,2\,R^2}{c_{12} + R} \end{aligned} \tag{11}$$

Thus

$$e_1 = \frac{T\,(c_{12} + R)}{R\,(3\,c_{12} + 2\,R)} \tag{12}$$

Hence

$$\begin{aligned} e_2 = e_3 &= \frac{-\,e_1\,c_{12}}{2\,(c_{12} + R)} \\ &= \left[\frac{-\,c_{12}}{2\,(c_{12} + R)}\right]\left[\frac{T\,(c_{12} + R)}{R\,(3\,c_{12} + 2\,R)}\right] \\ &= -\,\frac{c_{12}\,T}{2\,R\,(3\,c_{12} + 2\,R)} \end{aligned} \tag{13}$$

We define YOUNG'S MODULUS, E, as the simple longitudinal tension divided by the extension produced; thus

$$E = \frac{T}{e_1} = \frac{R\,(3\,c_{12} + 2\,R)}{c_{12} + R} \tag{14}$$

We define POISSON'S RATIO, σ, as the ratio of the lateral contraction to the longitudinal extension of a body under terminal tension; thus

$$\sigma = -\frac{e_3}{e_1} = -\frac{e_2}{e_1} = \left[\frac{c_{12}\,T}{2\,R\,(3\,c_{12} + 2\,R)}\right]\left[\frac{R\,(3\,c_{12} + 2\,R)}{T\,(c_{12} + R)}\right]$$

$$= \frac{c_{12}}{2\,(c_{12} + R)} \tag{15}$$

Whatever the stress system may be, the shearing strain corresponding with a pair of rectangular axes and the shearing stress on the pair of planes at right angles to those axes are given by

$$X_6 = R\,e_6,\ X_4 = R\,e_4,\ X_5 = R\,e_5 \tag{16}$$

and are independent of the directions of the axes. The quantity R is defined as the MODULUS OF RIGIDITY.

For convenience the relations between these elastic moduli for isotropic substances are summed up in the table on page 128.

84. Anisotropic character of homogeneous crystalline substances: The most important examples we have of non-isotropic homogeneous bodies are crystalline substances.

We assume as a physical hypothesis that the symmetry possessed by the crystallographic form of the substance applies also to every physical characteristic of the substance. The substance may, however, possess some physical characteristics that belong to a higher order of symmetry. An example of this is the optical isotropy of isometric crystals.

In general the stress-strain relations will be dependent on the rectangular set of coordinate axes chosen; however, the crystallographic symmetry relations make possible certain transformations of the coordinate axes for which the quadratic expression remains unaltered. The restrictions imposed on the strain components by

such transformations for which the quadratic expression remains unaltered result in a simplification of the elastic constants, the amount of simplification depending on the invariants of the transformation.[1]

ELASTIC MODULI OF ISOTROPIC SUBSTANCES

	$c_{11} = A$	$c_{12} = B$	R	$B = c_{12}$	K	R	E	σ
A	A	—	$B + 2R$		$\frac{3K + 4R}{3}$		$\frac{E(1-\sigma)}{1-\sigma-2\sigma^2}$	
B	—	B	—	B	$\frac{3K - 2R}{3}$		$\frac{E\sigma}{1-\sigma-2\sigma^2}$	
K	$\frac{c_{11} + 2c_{12}}{3}$		$B + \frac{2}{3}R$		K	—	$\frac{E}{3(1-2\sigma)}$	
R	$\frac{c_{11} - c_{12}}{2}$		R	—	—	R	$\frac{E}{2(1+\sigma)}$	
E	$\frac{(c_{11} - c_{12})(c_{11} + 2c_{12})}{c_{11} + c_{12}}$		$\frac{R(3B + 2R)}{B + R}$		$\frac{9KR}{3K + R}$		E	—
σ	$\frac{c_{12}}{c_{11} + c_{12}}$		$\frac{B}{2(R + B)}$		$\frac{3K - 2R}{2(3K + R)}$		—	σ
β	$\frac{3}{c_{11} + 2c_{12}}$		$\frac{3}{2R + 3B}$		$\frac{1}{K}$	—	$\frac{3(1-2\sigma)}{E}$	

In the following table crystalline substances have been arranged into classes and groups according to the kinds of symmetry the crystal form possesses[2] and the number of corresponding "fundamental elastic constants." Fundamental is used in the sense that these coefficients are all that are left after the above simpli-

[1] For the process by which this simplification is carried out see A. E. H. Love, A treatise on the mathematical theory of elasticity (Cambridge Univ. Press) 1920, pp. 149-158.

[2] The nomenclature of the systems and groups has been taken from Dana-Ford's Textbook of Mineralogy. For the symmetry relations of these systems and groups the reader is referred to this book.

fication has been carried out and thus no one of them can be obtained from a combination of the others. Below the table are listed the corresponding fundamental coefficients.

System	*Class*	*Example*	*Fundamental elastic constants*
Isometric	Normal	Galena	3
	Pyritohedral	Pyrite	3
	Tetrahedral	Tetrahedrite	3
	Plagiohedral	Cuprite	3
	Tetartohedral	Barium nitrate	3
Tetragonal	Normal	Zircon	6b
	Hemimorphic	Iodosuccinimide	6b
	Tripyramidal	Scheelite	7b
	Pyramidal-hemimorphic	Wulfenite	7b
	Sphenoidal	Chalcopyrite	6b
	Trapezohedral	Nickel sulphate	6b
	Tetartohedral	$2CaO.Al_2O_3.SiO_2$	7b
Hexagonal	Normal	Beryl	5
	Hemimorphic	Zincite	5
	Tripyramidal	Apatite	5
	Pyramidal-hemimorphic	Nephelite	5
	Trapezohedral	β-quartz	5
Trigonal or Rhombohedral	Trigonal	Benitoite	5
	Rhombohedral	Calcite	6a
	Rhombohedral-hemimorphic	Tourmaline	6a
	Tri-Rhombohedral	Phenacite	7a
	Trapezohedral	Quartz	6a
	Trigonal Bipyramidal	None	5
	Trigonal pyramidal	Sodium periodate	7a
Orthorhombic	Normal	Barite	9
	Hemimorphic	Calamine	9
	Sphenoidal	Epsomite	9
Monoclinic	Normal	Gypsum	13
	Hemimorphic	Tartaric acid	13
	Clinohedral	Clinohedrite	13
Triclinic	Normal	Axinite	21
	Asymmetric	Calcium thiosulphate	21

where

(21) denotes tha tall 21 coefficients are fundamental and thus none can be eliminated.

(13) denotes that $c_{11}, c_{12}, c_{13}, c_{16}, c_{22}, c_{23}, c_{26}, c_{33}, c_{36}, c_{44}, c_{45}, c_{55}, c_{66}$ are all fundamental, the others being identically zero.

Similarly
for (9) we have:

$$c_{11},\ c_{12},\ c_{13},\ c_{22},\ c_{23},\ c_{33},\ c_{44},\ c_{55},\ c_{66}$$

for (7a):

$$c_{11} = c_{22},\ c_{12},\ c_{13} = c_{23},\ c_{14} = -c_{24} = c_{56},$$

$$c_{15} = -c_{25} = -c_{46},\ c_{33},\ c_{44} = c_{55},\ c_{66} = \frac{1}{2}(c_{11} - c_{12})$$

for (6a):

$$c_{11} = c_{22},\ c_{12},\ c_{13} = c_{23},\ c_{15} = -c_{25} = -c_{46},\ c_{33},$$

$$c_{44} = c_{55},\ c_{66} = \frac{1}{2}(c_{11} - c_{12}).$$

for (5):

$$c_{11} = c_{22},\ c_{12},\ c_{13} = c_{23},\ c_{33},\ c_{44} = c_{55},\ c_{66} = \frac{1}{2}(c_{11} - c_{12})$$

for (7b):

$$c_{11} = c_{22},\ c_{12},\ c_{13} = c_{23},\ c_{16} = -c_{26},\ c_{33},\ c_{44} = c_{55},\ c_{66}.$$

for (6b):

$$c_{11} = c_{22},\ c_{12},\ c_{13} = c_{23},\ c_{33},\ c_{44} = c_{55},\ c_{66}.$$

and for (3):

$$c_{11} = c_{22} = c_{33},\ c_{12} = c_{13} = c_{23},\ c_{44} = c_{55} = c_{66}.$$

In the following discussion the stress-strain relations shall be developed for the generalized strain-energy function. Then if we desire the strain-energy function for a certain crystalline body we merely have to make the substitutions according to the table just given.

85. Transformations of the strain-energy function: Solving the equations,

$$X_i = \sum_{h=1}^{6} c_{ih}\, e_h,\ i = 1, \cdots, 6$$

of § 81 for e_1 we have

$$e_1 = \frac{\begin{vmatrix} X_1 & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ X_2 & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ X_3 & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ X_4 & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ X_5 & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ X_6 & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{vmatrix}}{\begin{vmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{vmatrix}}$$

Similarly we can solve for $e_2, \cdots, e_6$.

Now let J denote the determinant that corresponds with the denominator of the expression for e_1 and let J_{rs} denote the minor determinant that corresponds with c_{rs}, then

$$\begin{aligned} J\,e_1 &= X_1 J_{11} - X_2 J_{21} + X_3 J_{31} - X_4 J_{41} + X_5 J_{51} - X_6 J_{61} \\ J\,e_2 &= -X_1 J_{12} + X_2 J_{22} - X_3 J_{32} + X_4 J_{42} - X_5 J_{52} + X_6 J_{62} \\ &\cdot \\ &\cdot \\ &\cdot \\ J\,e_6 &= -X_1 J_{16} + X_2 J_{26} - X_3 J_{36} + X_4 J_{46} - X_5 J_{56} + X_6 J_{66} \end{aligned}$$

where $J_{rs} = J_{sr}$, $(s, r) = 1, \cdots, 6$.

The quantities $\frac{1}{2} c_{ii}$, c_{ih}, $(i, h) = 1, \cdots, 6, i \neq h$, are the coefficients of a homogeneous quadratic function of e_k, $k = 1, \cdots, 6$. This function is the strain-energy-function expressed in terms of the strain components. Likewise the quantities $\frac{1}{2}\frac{J_{ii}}{J}$, $\frac{J_{ih}}{J}$ are the

coefficients of a homogeneous quadratic function of X_k. This function is the strain-energy-function expressed in terms of stress components.

86. Moduli of elasticity for anisotropic homogeneous substances: Let

$$X_1 = X_2 = X_3 = -P$$
$$X_4 = X_5 = X_6 = 0$$

then the corresponding is the cubical dilatation:

$$\Delta = e_1 + e_2 + e_3$$

and, from § 85,

$$J\,e_1 = -P\,(J_{11} - J_{21} + J_{31})$$
$$J\,e_2 = -P\,(-J_{12} + J_{22} - J_{32})$$
$$J\,e_3 = -P\,(J_{13} - J_{23} + J_{33})$$

Thus we have, since $J_{rs} = J_{sr}$, $(s, r) = 1, \cdots, 6$,

$$-J\,\Delta = P\,(J_{11} + J_{22} + J_{33} + 2\,J_{13} - 2\,J_{12} - 2\,J_{23})$$

Hence the *bulk modulus or modulus of compression*, K, is

$$K = -\frac{P}{\Delta} = \frac{J}{J_{11} + J_{22} + J_{33} - 2\,J_{12} + 2\,J_{13} - 2\,J_{23}}$$

If we let $X_1 = X_2 = X_3 = X_5 = X_6 = 0$, and X_4 be the shearing stress applied, $J\,e_4 = X_4\,J_{44}$; then R_{yz}, the *modulus of rigidity* corresponding to the pair of directions y, z, is $\frac{J}{J_{44}}$.

Similarly for the rigidity corresponding to other pairs of directions.

If we let $X_2 = X_3 = X_4 = X_5 = X_6 = 0$, then

$$J\,e_1 = X_1\,J_{11}$$
$$J\,e_2 = X_1\,J_{12}$$
$$J\,e_3 = X_1\,J_{13}$$

and thus E_x, *Young's Modulus* corresponding to the direction x is $\dfrac{J}{J_{11}}$.

Similarly for Young's moduli corresponding to other directions.

Poisson's ratio in the y direction is then

$$-\frac{e_2}{e_1} = -\frac{\left(\dfrac{J_{12}}{J}\right)X_1}{\left(\dfrac{J_{11}}{J}\right)X_1} = -\frac{J_{12}}{J_{11}},$$

and in the z direction is

$$-\frac{e_3}{e_1} = -\frac{J_{13}}{J_{11}}.$$

The value of Poisson's ratio for anisotropic substances thus depends on the direction of the contracted linear elements as well as on the direction of the extended longitudinal ones.

CHAPTER IX

Systems not in equilibrium. Irreversible processes

87. Displacement transformations: If the system is not in equilibrium under the action of the external forces it will be moving from one configuration to another. Each of these configurations can be represented by its displacements.

PRINCIPLE OF SUPERPOSITION

We assume as a physical hypothesis that each component stress is accompanied by the same strains whether it acts alone or in conjunction with other stresses.

Now the body in motion will possess kinetic energy which depends on the distribution of mass and velocity. We shall consider the case of very small displacements (see §60 for definition).

Let u, v, w be the coefficients of the displacement by which the body passes from its state of reference to the strained state.

Assume that the kinetic energy per unit volume can be expressed by

$$\frac{1}{2}\rho\left[\left(\frac{\partial u}{\partial s}\right)^2+\left(\frac{\partial v}{\partial s}\right)^2+\left(\frac{\partial w}{\partial s}\right)^2\right] \tag{1}$$

where ρ is the density of the body in the state of reference and u, v, w are functions of x, y, z, s; x, y, z denoting the coordinates and s the time.

The rate at which work is done by the body forces is

$$I_1 = \iiint_v \rho\left[X\frac{\partial u}{\partial s}+Y\frac{\partial v}{\partial s}+Z\frac{\partial w}{\partial s}\right]dv \tag{2}$$

where v represents the volume of the body in the state of reference.

The rate at which work is done by the surface tractions is

$$\iint_\sigma \left[X_\nu\frac{\partial u}{\partial s}+Y_\nu\frac{\partial v}{\partial s}+Z_\nu\frac{\partial w}{\partial s}\right]d\sigma \tag{3}$$

where σ represents the surface of the body in the state of reference.

Transforming by Green's Theorem and using the relations of (65.1, 2, 3) and of (65.8), and the notation of (60.1) for $e_1, \cdots,$ e_6, we have, for the rate at which work is done by the surface tractions

$$I_2 = \iiint\limits_{v} \left[\left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial Z_x}{\partial z} \right) \frac{\partial u}{\partial s} + \left(\frac{\partial X_y}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} \right) \frac{\partial v}{\partial s} + \left(\frac{\partial Z_x}{\partial x} + \frac{\partial Y_z}{\partial y} + \frac{\partial Z_z}{\partial z} \right) \frac{\partial w}{\partial s} \right] dv + \iiint\limits_{v} \left[X_x \frac{\partial e_1}{\partial s} + Y_y \frac{\partial e_2}{\partial s} + Z_z \frac{\partial e_3}{\partial s} + Y_z \frac{\partial e_4}{\partial s} + Z_x \frac{\partial e_5}{\partial s} + X_y \frac{\partial e_6}{\partial s} \right] dv \tag{4}$$

where v represents the volume of the body in the state of reference.

The rate of increase of kinetic energy, obtained by differentiating (87.1) with respect to s, is

$$\iiint\limits_{v} \rho \left(\frac{\partial^2 u}{\partial s^2} \frac{\partial u}{\partial s} + \frac{\partial^2 v}{\partial s^2} \frac{\partial v}{\partial s} + \frac{\partial^2 w}{\partial s^2} \frac{\partial w}{\partial s} \right) dv \tag{5}$$

Substituting the equations of motion, *i.e.*

$$\begin{aligned} \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + \rho X &= \rho \frac{\partial^2 u}{\partial s^2} \\ \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} + \rho Y &= \rho \frac{\partial^2 v}{\partial s^2} \\ \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} + \rho Z &= \rho \frac{\partial^2 w}{\partial s^2} \end{aligned} \tag{6}$$

in (87.5) we have

$$I_3 = \iiint\limits_{v} \left[\left(\rho X + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} \right) \frac{\partial u}{\partial s} + \left(\rho Y + \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} \right) \frac{\partial v}{\partial s} + \left(\rho Z + \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} \right) \frac{\partial w}{\partial s} \right] dx\, dy\, dz \tag{7}$$

Thus we have

$$\mathrm{I}_1 + \mathrm{I}_2 - \mathrm{I}_3 = \iiint\limits_{v} \left[X_x \frac{\partial e_1}{\partial s} + Y_y \frac{\partial e_2}{\partial s} + Z_z \frac{\partial e_3}{\partial s} + X_y \frac{\partial e_6}{\partial s} + Y_z \frac{\partial e_4}{\partial s} + Z_x \frac{\partial e_5}{\partial s} \right] dv \tag{8}$$

which is the excess of the rate at which work is done by the external forces above the rate of increase of the kinetic energy, *i.e.* gives us the rate of increase in internal energy due to deformation.

88. Definitions of work and heat received: Thus the work, **W**, done on the system is given by the equation

$$\begin{aligned} \mathbf{W} = \int_{s_0}^{s}\!\!\iiint\limits_{v} \Big\{ & \left[\rho X + \frac{\partial X_1}{\partial x} + \frac{\partial X_6}{\partial y} + \frac{\partial X_5}{\partial z} \right] \frac{\partial u}{\partial s} + \\ & \left[\rho Y + \frac{\partial X_6}{\partial x} + \frac{\partial X_2}{\partial y} + \frac{\partial X_4}{\partial z} \right] \frac{\partial v}{\partial s} + \\ & \left[\rho Z + \frac{\partial X_5}{\partial x} + \frac{\partial X_4}{\partial y} + \frac{\partial X_3}{\partial z} \right] \frac{\partial w}{\partial s} \Big\} dv\, ds + \\ \int_{s_0}^{s}\!\!\iiint\limits_{v} & \left\{ X_1 \frac{\partial e_1}{\partial s} + \cdots + X_6 \frac{\partial e_6}{\partial s} \right\} dv\, ds. \end{aligned} \tag{1}$$

where $X_1, \cdots, X_6$, are functions of $e_1, \cdots, e_6, t$; $e_1, \cdots, e_6, t$ are functions of x, y, z, and the time s. **v** represents the volume of the body in the state of reference.

Similarly we have for the heat, **Q**, received by the body

$$\begin{aligned} \mathbf{Q} = \int_{s_0}^{s}\!\!\iiint\limits_{v} \Big[& \left(\frac{\partial l_{e_1}}{\partial x} + \frac{\partial l_{e_6}}{\partial y} + \frac{\partial l_{e_5}}{\partial z} \right) \frac{\partial u}{\partial s} + \left(\frac{\partial l_{e_6}}{\partial x} + \frac{\partial l_{e_2}}{\partial y} + \frac{\partial l_{e_4}}{\partial z} \right) \frac{\partial v}{\partial s} + \\ & \left(\frac{\partial l_{e_5}}{\partial x} + \frac{\partial l_{e_4}}{\partial y} + \frac{\partial l_{e_3}}{\partial z} \right) \frac{\partial w}{\partial s} \Big] dv\, ds + \\ \int_{s_0}^{s}\!\!\iiint\limits_{v} & \left[c_e \frac{\partial t}{\partial s} + l_{e_1} \frac{\partial e_1}{\partial s} + \cdots + l_{e_6} \frac{\partial e_6}{\partial s} \right] dv\, ds. \end{aligned} \tag{2}$$

where c_e, l_{e_1}, $\cdots$, l_{e_6} are functions of $e_1, \cdots, e_6, t$; $e_1, \cdots, e_6, t$ are functions of x, y, z and the time s. v denotes the volume of the body in the state of reference.

Since a force always acts equally in opposite directions when the point of application moves, an amount of work is received by one system and an equal amount given up by the other system.

Similarly for heat the amount received by one system is equal to the amount given up by the other system.[1]

As we readily perceive, quantity of heat is a fundamental concept of thermodynamics. Heat is, however, a term so universally applied by all of us that we may be inclined to consider it as a rather straightforward concept given immediately in terms of everyday experience, but an analysis of the operation by which we measure quantity of heat will show us that the situation is not simple but extremely complicated. In fact no physical significance can be given directly to heat flow because there are no operations by which we can measure it as such. All we can measure are temperature changes and distributions and rates of increase or decrease of temperature.

One of the justifications for the treatment of heat as a quantity is that if two systems A and B are in contact and both otherwise isolated from the rest of the universe, no work being received by either of the systems, and if A then undergoes a change α while B undergoes a change β the heat received by A is defined as an integral which we shall call I and that lost by B is defined by a second integral, II, made equal to the first, I = II. Secondly if B and C are in contact and both otherwise isolated from the rest of the universe and if A undergoes a change α while C undergoes a change γ the heat received by A has been defined by integral I and that lost by C is defined by a third integral, III, made equal to I, I = III. Now if B and C are in contact but otherwise isolated from the rest of the universe and if B then undergoes a change β then our physical hypothesis is that C will undergo a change γ.

89. The first law of thermodynamics: Let K represent the kinetic and ϵ the internal energy of the system per unit mass.

[1] The statement that the energy of an isolated system is constant is not the first law of thermodynamics.

Then the first law of thermodynamics is expressed by the equation

$$\iiint_{\mathrm{v}} (\epsilon_1 + K_1)\,\rho\,\mathrm{dv} - \iiint_{\mathrm{v}} (\epsilon_0 + K_0)\,\rho\,\mathrm{dv} =$$

$$\int_{\mathrm{s}_0}^{\mathrm{s}}\iiint_{\mathrm{v}} \left[c_e \frac{\partial t}{\partial \mathrm{s}} + l_{e_1}\frac{\partial e_1}{\partial \mathrm{s}} + \cdots + l_{e_6}\frac{\partial e_6}{\partial \mathrm{s}}\right] \mathrm{dv\,ds} +$$

$$\int_{\mathrm{s}_0}^{\mathrm{s}}\iiint_{\mathrm{v}} \left[X_1 \frac{\partial e_1}{\partial \mathrm{s}} + \cdots + X_6 \frac{\partial e_6}{\partial \mathrm{s}}\right] \mathrm{dv\,ds} +$$

$$\int_{\mathrm{s}_0}^{\mathrm{s}}\iiint_{\mathrm{v}} \left\{\left[\frac{\partial l_{e_1}}{\partial \mathrm{x}} + \frac{\partial l_{e_6}}{\partial \mathrm{y}} + \frac{\partial l_{e_5}}{\partial \mathrm{z}}\right]\frac{\partial \mathrm{u}}{\partial \mathrm{s}} + \left[\frac{\partial l_{e_6}}{\partial \mathrm{x}} + \frac{\partial l_{e_2}}{\partial \mathrm{y}} + \frac{\partial l_{e_4}}{\partial \mathrm{z}}\right]\frac{\partial \mathrm{v}}{\partial \mathrm{s}} + \right.$$

$$\left.\left[\frac{\partial l_{e_5}}{\partial \mathrm{x}} + \frac{\partial l_{e_4}}{\partial \mathrm{y}} + \frac{\partial l_{e_3}}{\partial \mathrm{z}}\right]\frac{\partial \mathrm{w}}{\partial \mathrm{s}}\right\} \mathrm{dv\,ds} +$$

$$\int_{\mathrm{s}_0}^{\mathrm{s}}\iiint_{\mathrm{v}} \left\{\left[\rho X + \frac{\partial X_1}{\partial \mathrm{x}} + \frac{\partial X_6}{\partial \mathrm{y}} + \frac{\partial X_5}{\partial \mathrm{z}}\right]\frac{\partial \mathrm{u}}{\partial \mathrm{s}} + \right.$$

$$\left[\rho Y + \frac{\partial X_6}{\partial \mathrm{x}} + \frac{\partial X_2}{\partial \mathrm{y}} + \frac{\partial X_4}{\partial \mathrm{z}}\right]\frac{\partial \mathrm{v}}{\partial \mathrm{s}} +$$

$$\left.\left[\rho Z + \frac{\partial X_5}{\partial \mathrm{x}} + \frac{\partial X_4}{\partial \mathrm{y}} + \frac{\partial X_3}{\partial \mathrm{z}}\right]\frac{\partial \mathrm{w}}{\partial \mathrm{s}}\right\} \mathrm{dv\,ds}$$

where v denotes the volume and ρ the density of the body in the state of reference.

90. The second law of thermodynamics: The second law of thermodynamics, where η denotes the entropy of the system per unit mass, is expressed by the inequality

$$\iiint_{\mathrm{v}} \eta_1\,\rho\,\mathrm{dv} - \iiint_{\mathrm{v}} \eta_0\,\rho\,\mathrm{dv} >$$

$$\int_{s_0}^{s}\iiint_{v}\frac{1}{\theta}\left[c_e\frac{\partial t}{\partial s}+l_{e_1}\frac{\partial e_1}{\partial s}+\cdots+l_{e_6}\frac{\partial e_6}{\partial s}\right]dv\,ds+$$

$$\int_{s_0}^{s}\iiint_{v}\frac{1}{\theta}\left\{\left[\frac{\partial l_{e_1}}{\partial x}+\frac{\partial l_{e_6}}{\partial y}+\frac{\partial l_{e_5}}{\partial z}\right]\frac{\partial u}{\partial s}+\left[\frac{\partial l_{e_6}}{\partial x}+\frac{\partial l_{e_2}}{\partial y}+\frac{\partial l_{e_4}}{\partial z}\right]\frac{\partial v}{\partial s}+\right.$$

$$\left.\left[\frac{\partial l_{e_5}}{\partial x}+\frac{\partial l_{e_4}}{\partial y}+\frac{\partial l_{e_3}}{\partial z}\right]\frac{\partial w}{\partial s}\right\}dv\,ds$$

where $\mathbf{v}$ denotes the volume and ρ the density of the body in the state of reference.

91. Relations of non-homogeneous to homogeneous systems: By definition a system is said to be in a state of equilibrium if the properties of the system by which the state is defined undergo no change after the lapse of a period of time no matter how greatly it is extended. Hence in an equilibrium state the properties defining the state of the system are constants with respect to time.

Now we can set up a system acted on by external fields of force such that the stresses would be functions of the coordinates of the system and yet constant with respect to time. Such fields of forces we call body forces and the most common example of such a field of force is the gravitational field of the Earth.

The work received by the system from the gravitational field, if we assume gravity acts in the negative direction of the z-axis, will be

$$\int_{z_0}^{z}\iiint_{v} g\,\rho\,dv\,dz$$

where g denotes the force of gravity, ρ and $\mathbf{v}$ the density and volume respectively of the system in the state of reference.

But a necessary assumption we make is that such variable properties be continuous functions of the coordinates of the

system. This being the case it is possible to choose a neighborhood about any point of the system such that in this neighborhood the properties of the system will differ from the properties at the point about which the neighborhood was drawn by less than any previously assigned small positive quantity K. In other words, the system in this neighborhood will remain sensibly homogeneous.

Hence we can divide the system up into a finite number of neighborhoods or regions each of which is sensibly homogeneous and thus treat such a system as if it were made up of a finite number of homogeneous parts.

Now if we expand our neighborhood to include the whole system which we then define as a homogeneous system, $t, e_1, \cdots, e_6$ will be constants with respect to the coordinates of the system for any one state.

Hence as $t, e_1, \cdots, e_6$ approach constant values with respect to time and the coordinates x, y, z for any one state the expression for the work received by the system approaches as a limit the integral

$$\mathrm{v}\int_{s_0}^{s} \sum_{i=1}^{6} X_i \frac{de_i}{ds}\, ds$$

where $e_1, \cdots, e_6$ depend upon the path s, **v** denoting the total volume of the system in the reference state.

Similarly the expression for the heat received approaches as a limit the integral

$$\mathrm{v}\int_{s_0}^{s} \left[c_e \frac{dt}{ds} + \sum_{i=1}^{6} l_{e_i} \frac{de_i}{ds} \right] ds$$

where $c_e, l_{e_1}, \cdots, l_{e_6}$ are functions of $t, e_1, \cdots, e_6$; and $t, e_1, \cdots, e_6$ depend on the path s; **v** is the volume of the system in the state of reference.

Now as $\frac{\partial u}{\partial s}, \frac{\partial v}{\partial s}$, and $\frac{\partial w}{\partial s}$ simultaneously approach zero the change in kinetic energy approaches zero as a limit and we have

$$\lim_{\left(\frac{\partial u}{\partial s}, \frac{\partial v}{\partial s}, \frac{\partial w}{\partial s}\right) \to 0} \iiint_{\mathrm{v}} (K_1 - K_0)\, \rho\, d\mathrm{v} = 0$$

K being a function of the density and velocity only.

Hence since $\frac{\varepsilon}{\mathrm{v}}$ and its partial derivatives are by hypothesis continuous functions of $t, e_1, \cdots, e_6$ only, $\frac{\varepsilon}{\mathrm{v}}(t, e_1, \cdots, e_6)$, the first law of thermodynamics reduces to

$$\frac{\varepsilon}{\mathrm{v}}(t, e_1, \cdots, e_6) - \frac{\varepsilon}{\mathrm{v}}(t_0, e_{1_0}, \cdots, e_{6_0})$$

$$\int_{t_0, e_{1_0}, \ldots, e_{6_0}}^{t, e_1, \ldots, e_6} c_e\, dt + \sum_{i=1}^{6} [l_{e_i} + X_i]\, de_i$$

where v is the volume in the state of reference.

If the properties defining the state reduce to temperature and volume the first law can be expressed by the equation

$$\varepsilon(t, \mathrm{v}) - \varepsilon(t_0, \mathrm{v}_0) = \mathrm{m} \int_{t_0, v_0}^{t, v} c_v\, dt + (l_v - p)\, dv$$

where c_v and l_v are functions of temperature and the specific volume, $v = \frac{\mathrm{v}}{\mathrm{m}}$.

We assume as a physical hypothesis that as the series of states through which the system passes in a process become more and more nearly equal to a series of equilibrium states the total entropy of the system approaches as a limit the value of the integral

$$\mathrm{v} \int_{t_0, e_{1_0}, \ldots, e_{6_0}}^{t, e_1, \ldots, e_6} \frac{c_e}{\theta}\, dt + \sum_{i=1}^{6} \frac{l_{e_i}}{\theta}\, de_i$$

where θ, a function of t only, is the same for all systems, and $\frac{\mathrm{n}}{\mathrm{v}}$ with

its partial derivatives are continuous functions of $t, e_1, \cdots, e_6$ only, $\mathbf{v}$ denoting the volume in the state of reference.

Thus

$$\frac{\mathbf{n}}{\mathbf{v}}(t, e_1, \cdots, e_6) - \frac{\mathbf{n}}{\mathbf{v}}(t_0, e_{1_0}, \cdots, e_{6_0}) = \int_{t_0, e_{1_0}, \ldots, e_{6_0}}^{t, e_1, \ldots, e_6} \frac{c_e}{\theta}\,\mathrm{d}t + \sum_{i=1}^{6} \frac{l_{e_i}}{\theta}\,\mathrm{d}e_i$$

If the properties defining the state reduce to temperature and volume the second law can be expressed by the equation

$$\mathbf{n}(t, \mathbf{v}) - \mathbf{n}(t_0, \mathbf{v}_0) = \mathbf{m} \int_{t_0, v_0}^{t, v} \frac{c_v}{\theta}\,\mathrm{d}t + \frac{l_v}{\theta}\,\mathrm{d}v$$

where c_v and l_v are functions of temperature and the specific volume $v = \dfrac{\mathbf{v}}{\mathbf{m}}$.

92. "Reversible Processes:" Now in treating homogeneous systems we speak of a series of equilibrium states in which each property of the system varies continuously as a continuous series of states. But since, by definition, the system can not pass through a continuous series of equilibrium states it would be absurd to speak of "the work or heat received by the system in passing through a continuous series of equilibrium states."

Now we have defined a quantity $\mathbf{W}$ for the continuous series of equilibrium states

$$\mathbf{W} = \mathbf{v} \int_{t_0, e_{1_0}, \ldots, e_{6_0}}^{t, e_1, \ldots, e_6} \sum_{i=1}^{6} X_i\,\mathrm{d}e_i$$

and this is the limit of the expression for work in a process as the series of states the system goes through approaches a series of equilibrium states.

Similarly we have defined a quantity $\mathbf{Q}$ for the continuous series of equilibrium states

$$\mathbf{Q} = \mathbf{v} \int_{t_0, e_{1_0}, \ldots, e_{6_0}}^{t, e_1, \ldots, e_6} c_e\,\mathrm{d}t + \sum_{i=1}^{6} l_{e_i}\,\mathrm{d}e_i$$

Now we can make our process go through a series of states which differ from a series of equilibrium states by an amount that is less than some previously assigned small positive quantity δ, and thus have our process approach as near as we please to a series of equilibrium states. This is what we mean to imply when we speak of a reversible process.

Such a series of states has also been called a quasi-static process and such states quasi-static states.

Hence **Q** for a continuous series of equilibrium states is determined by the limit that **Q** for a continuous series of non-equilbrium states approaches as these states become more and more nearly equal to the equilibrium states. Similarly for **W** of a continuous series of equilibrium states.

Thus it is the physics of the situation that demands the treatment of systems not in equilibrium should logically precede the treatment of systems in equilibrium.

93. Justification for defining the entropy and energy of a simple substance at some arbitrary state as zero: Now we have seen how we may obtain the difference in energy and entropy of a system between any two equilibrium states. Furthermore it is only differences in energy and entropy that we can measure and therefore that have a physical meaning in classical thermodynamics. As Gibbs[1] says, "the values of these quantities are so far arbitrary, that we may choose independently for each simple substance the state in which its energy and entropy are both zero. The values of the energy and entropy of any compound body will then be fixed." Furthermore, the state in which the entropy of the simple substance is chosen as zero need not be the state in which its energy is chosen as zero.

Therefore we can define the energy and entropy of a simple substance as zero at some convenient state.

However, suppose we wish to coordinate thermodynamics with other fields of science.

Now when a moving particle is acted on by a conservative[2] force

[1] J. Willard Gibbs, Collected Works, vol. 1, p. 85.

[2] A conservative force is one such that any work done by displacing a system against it would be completely regained if the motion of the system was reversed.

we assume as a physical hypothesis that its kinetic energy has been transformed into potential energy. The increase in the potential energy of the particle is then equal to the kinetic energy which has been destroyed and hence equal to the work done by the particle against the force.

When an isolated particle is set in motion we have from the theory of relativity

$$\mathbf{K}_1 - \mathbf{K}_0 = \int c^2 \, dm$$

where c denotes the velocity of light and **m** the mass of the particle, and at zero velocity $\mathbf{K}_0$ is defined as zero. Thus the kinetic energy of the particle in ergs is equal to the mass in grams multiplied by the square of the velocity of light. Furthermore, Tolman[1] says

"when a moving particle is brought to rest and thus loses both its kinetic energy and its extra ('kinetic') mass, there seems to be every reason for believing that this mass and energy which are associated together when the particle is in motion and leave the particle when it is brought to rest will still remain always associated together. For example, if the particle is brought to rest by collision with another particle, it is an evident consequence of our considerations that the energy and the mass corresponding to it do remain associated together since they are both passed on to the new particle. On the other hand, if the particle is brought to rest by the action of a conservative force, say for example that exerted by an elastic spring, the kinetic energy which leaves the particle will be transformed into the potential energy of the stretched spring, and since the mass which has undoubtedly left the particle must still be in existence, we shall believe that this mass is now associated with the potential energy of the stretched spring.

"Such considerations have led us to believe that matter and energy may be best regarded as different names for the same fundamental entity: *matter*, the name which has been applied

[1] R. C. Tolman, The Theory of the Relativity of Motion (Univ. of Calif. Press) 1917, pp. 83, 84.

when we have been interested in the property of mass or inertia possessed by the entity, and *energy*, the name applied when we have been interested in the part taken by the entity in the production of motion and other changes in the physical universe."

But from the equation above we have for one gram of matter

$$\text{Energy} = c^2 = (2.9986 \times 10^{10})^2 = \text{approx. } 9 \times 10^{20} \text{ ergs}$$

Hence to measure the energy change to an accuracy of plus or minus one erg the change in mass must be measured to an accuracy of plus or minus 10^{-21} grams. Therefore, except for systems such as electrons moving at high velocities, this method of measuring energy is of little value to us.

Again we know that the loss in energy from say room temperature to absolute zero on the thermodynamic scale of temperature will be inappreciable compared with its total energy. However, if we wish to define energy in conformity with the theory of relativity we could define the energy of the system at some arbitrary state $(t_0, e_{1_0}, \cdots, e_{6_0})$ as mc^2 and then use our ordinary methods for obtaining the energy change between this and any other state. The accuracy with which the energy is determined will depend then on the accuracy with which the mass was determined at the state $(t_0, e_{1_0}, \cdots, e_{6_0})$ and the accuracy of our measurements of the heat and work of a process. We assume that for our measurements of the heat and work we are dealing only with systems for which the velocity of the process can be made as small as we please and that we are dealing only with local space.

CHAPTER X.

Introduction to the Tables of Thermodynamic Relations

94. The variable properties or quantities of the tables and their relations: In the following tables thermodynamic relationships between the derivatives of the following $10 + n$ variable properties or quantities are given.

θ = temperature in degrees on the absolute thermodynamic scale. ($\theta_s - \theta_i = 100°$ where θ_s denotes the temperature of steam and θ_i that of ice at one atmosphere pressure.)

p = pressure in dynes per square centimeter or baryes.

$\mathbf{m}_k$ = mass in grams of component k, $k = 1, \cdots, n$.

$\mathbf{v}$ = volume in cubic centimeters of the system (phase).

ε = energy in dyne centimeters of the system (phase).

$\mathfrak{n}$ = entropy in dyne centimeters per degree of the system (phase).

$\zeta = \varepsilon + p\mathbf{v} - \theta\mathfrak{n}$

$\chi = \varepsilon + p\mathbf{v}$

$\psi = \varepsilon - \theta\mathfrak{n}$

$\mathbf{W}$ = work, in dyne centimeters, received by the system.

$\mathbf{Q}$ = heat, in dyne centimeters, received by the system.

The relations between the $10 + n$ quantities, with their first and second derivatives, are connected by various relations. The relations between the quantities themselves are rather simple and chiefly of the nature of definitions and hypotheses. But the relations between the first and second derivatives are more complicated and it is these in which we are now interested.

Let us consider the first derivatives. The physical hypotheses made, which include the characteristic equation of the system and the first and second laws of thermodynamics together with the definitions of the secondary quantities ζ, χ, ψ, give us the functional

relations necessary for the existence and continuity of the derivatives.

In each derivative of the type

$$\left(\frac{\partial \varepsilon}{\partial v}\right)_{\theta,\, p,\, m_2,\, \ldots,\, m_n}$$

there are n + 3 independent variables, n + 1 of them being held fast. If we then separate these derivatives into groups according to the n + 1 variables which are held fast, we shall have $\frac{(n+8)!}{(n+1)!\,7!}$ groups in the table of first derivatives and 42 derivatives in each group. Furthermore there will be 14 derivatives of the type $\frac{dW}{dv}$, where $W = f(S)$, $v = S$, $\theta = K'$, $p = K''$, $m_i = K_i'''$, $i = 2, \cdots, n$ K', K'', K_i''' being constants, in each group.

This makes a total of $\frac{(n+8)!}{(n+1)!\,90}$ first derivatives.

Now these $\frac{(n+8)!}{(n+1)!\,90}$ derivatives, 18,480 for a ternary system, are connected by various functional relations, and in general there is an equation connecting any 4 +3n of them and certain of the 10 + n variable properties. There are therefore

$$\frac{\left[\dfrac{(n+8)!}{(n+1)!\,90}\right]!}{(4+3n)!\left[\dfrac{(n+8)!}{(n+1)!\,90}-(4+3n)\right]!}$$

such relations between the first derivatives. For a ternary system this would be $\frac{18480!}{13!\,18467!}$.

Such a tabulation is out of the question. However, a table can be derived in which the $\frac{(n+8)!}{(n+1)!\,90}$ derivatives can be obtained in terms of the same 3n + 3 standard derivatives. Then to obtain any desired one of the numerous relations between any 3n + 4 of

the $\frac{(n+8)\,!}{(n+1)\,!\,90}$ derivatives in Table I we merely have to eliminate the $3n + 3$ standard derivatives between the $3n + 4$ equations for the $3n + 4$ derivatives. It should be remarked that the method used here of tabulating the $\frac{(n+8)\,!}{(n+1)\,!\,90}$ derivatives in terms of the same standard $3n + 3$ will largely do away with the necessity for determining the other relations by an elimination. For if in any special problem every quantity of interest is kept in terms of the same standard $3n + 3$ one may be sure that at the end of the discussion there are no essential relations not brought to light.

95. The Standard Derivatives for Table I: The $3n + 3$ standard derivatives may be chosen in many ways. The $3n + 3$ chosen here are

the isothermal compressibility, $\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_1,\, \ldots,\, m_n}$;

the isopiestic dilatation, $\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_1,\, \ldots,\, m_n}$;

the change of volume with change in mass of component K, $\left(\frac{\partial v}{\partial m_k}\right)_{\theta,\, p,\, m_i}$, m_i denoting all the component masses except m_k;

the heat capacity at constant pressure, $(m_1 + \cdots + m_n)\, c_p$; the heat of change of mass of component k at constant temperature and pressure, l_{m_k}, $k = 1, \cdots, n$; and μ_k, $k = 1, \cdots, n$.

These functions tabulated as the standard derivatives are not fundamental in the sense that they are the minimum number of quantities left after elimination by means of all available quantities. However, if it is desired to express the $\frac{(n+8)\,!}{(n+1)\,!\,90}$ derivatives of the tables in terms of the fundamental $3n$ derivatives this can be done by means of the transformations of Table A in which the standard derivatives are expressed in terms of the fundamental derivatives, which are intensive quantities, and the masses.

96. Transformations of Standard Derivatives of Table I to Fundamental Derivatives:

TABLE A.

$$\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n} = (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_1,\, \ldots,\, m_{n-1}}$$

$$\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_n} = (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_1,\, \ldots,\, m_{n-1}}$$

$$\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}\right)_{\theta,\, p,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_{k-1},\, \mathrm{m}_{k+1},\, \ldots,\, \mathrm{m}_n} = v + \left(\frac{\partial v}{\partial m_k}\right)_{\theta,\, p,\, m_1,\, \ldots,\, m_{k-1},\, m_{k+1},\, \ldots,\, m_{n-1}} - \sum_{i=1}^{n-1} m_i \frac{\partial v}{\partial m_i}, \text{ if } k \neq n$$

$$= v - \sum_{i=1}^{n-1} m_i \frac{\partial v}{\partial m_i}, \text{ if } k = n$$

$$\mathrm{l}_{m_k} = \theta\,\eta + (l_{m_k} - l_{m_n}) - \sum m_i\,(l_{m_i} - l_{m_n})$$

$$\mu_k = \epsilon + pv - \theta\,\eta + (\mu_k - \mu_n) - \sum m_i\,(\mu_i - \mu_n)$$

These 3n fundamental derivatives have been chosen rather than some other set of 3n derivatives because they are basic intensive quantities in the development of the theory of thermodynamics and, as will be shown later, are readily obtainable from experiment.

97. Abbreviations and Special Notation introduced in the Tables: Now it is possible, by introducing some abbreviations and special notations, which do not sacrifice clarity by so doing, to reduce the number of tabulations still further.

Since all the partial derivatives with respect to the component masses are symmetrical it is only necessary to write the derivative

with respect to one of the masses, the others being obtainable by symmetry. Thus

$$\left(\frac{\partial \mathbf{n}}{\partial \mathbf{m}_k}\right)_{\theta,\, p,\, m_1,\, \ldots,\, m_{k-1},\, m_{k+1},\, \ldots,\, m_n} = \frac{l_{m_k}}{\theta}$$

where k can have the value $1 \leqq k \leqq n$, k being an integer.

Furthermore, instead of listing each derivative separately, functions of temperature, pressure, and the masses or mass fractions are assigned to each of the $n + 10$ variable quantities such that when the quotient of the two corresponding functions is taken it will express the derivative. At first glance this might seem as though we had doubled the size of the tables, but we find the functional relationships to be such that each variable quantity needs to be tabulated only once in each group. Thus it is necessary to tabulate only 9 of these functions in each group, making a total of $\frac{(n + 8)!\,9}{(n + 1)!\,7!}$, for Table I of the first derivatives.

Hence the quotient of any two of the *same* group expresses the derivative, the variables held constant being given at the top of each group.

Thus in the first group

$$\frac{(\partial \mathrm{v})_{①}}{(\partial \mathbf{m}_1)_{①}} \equiv \left(\frac{\partial \mathrm{v}}{\partial \mathbf{m}_1}\right)_{\theta,\, p,\, m_2,\, \ldots,\, m_n}$$

and

$$\frac{(\mathrm{d}\mathbf{W})_{①}}{(\partial \mathbf{m}_1)_{①}} \equiv \frac{\mathrm{d}\mathbf{W}}{\mathrm{d}\mathbf{m}_1}, \text{ where } \mathbf{W} = f(S),\ \theta = K',\ p = K'',\ \mathbf{m}_1 = S,$$

$$\mathbf{m}_i = K_i''' \text{ where } i = 2, \cdots, n,$$

$$K', K'', K_i''' \text{ being constants.}$$

$$\equiv \left(\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}\mathbf{m}_1}\right)_{\theta,\, p,\, m_2,\, \ldots,\, m_n}$$

This notation was first used by Bridgman in 1914[1] in listing derivatives for the single phase one component system of constant mass. His table of first derivatives was reproduced by Lewis and

[1] P. W. Bridgman, Physical Review, 1914.

Randall.[1] Bridgman republished[2] his tables, correcting the typographical errors, and added a table for two phase one component systems of constant mass. It is therefore thought that this notation will be familiar to most readers. However, in order to avoid any misapprehension of this notation, it might be well to state that these expressions are not increments nor partial differentials but merely have such functional relationships that if the quotient is taken of any two in the same group this quotient expresses the desired derivative.

Table I contains all the first derivatives thought to be of practical value.

98. Transformations necessary to convert the tables for variable mass systems to tables for unit mass systems: The $10 + \mathrm{n}$ quantities of an n-component variable mass system, of which all except θ and p are extensive quantities, become $10 + \mathrm{n} - 1 = 9 + \mathrm{n}$ intensive quantities in an n-component unit mass system.

The reason we have one less variable in the unit mass system is because only $\mathrm{n} - 1$ of the mass fractions are independent, the nth mass fraction being given by the equation

$$m_1 + \ldots\ldots + m_{\mathrm{n}} = 1.$$

Thus to make Tables I and II applicable to systems of unit mass we must make the following substitutions in the tables: $\mathbf{v}, \boldsymbol{\varepsilon}, \mathbf{n}, \boldsymbol{\zeta}, \boldsymbol{\chi}, \boldsymbol{\psi}, \mathbf{W}, \mathbf{Q}$ replaced respectively by $v, \epsilon, \eta, \zeta, \chi, \psi, W, Q$; $\mathbf{m}_1, \cdots, \mathbf{m}_{\mathrm{n}}$ replaced by $m_1, \cdots, m_{\mathrm{n}-1}$; μ_1 by $\mu_1 - \mu_{\mathrm{n}}$, and l_{m_1} by $l_{m_1} - l_{m_{\mathrm{n}}}$, where 1 can be replaced by $2, \cdots, \mathrm{n} - 1$; and $(\mathbf{m}_1 + \cdots + \mathbf{m}_{\mathrm{n}})\, c_p$ by c_p.

Thus

$$\frac{(\partial\epsilon)_{①}}{(\partial m_1)_{①}} \equiv \left(\frac{\partial\epsilon}{\partial m_1}\right)_{\theta,\, p,\, m_2,\, \ldots,\, m_{\mathrm{n}-1}}$$

$$= (\mu_1 - \mu_{\mathrm{n}}) + (l_{m_1} - l_{m_{\mathrm{n}}}) - p\left(\frac{\partial v}{\partial m_1}\right)_{\theta,\, p,\, m_2,\, \ldots,\, m_{\mathrm{n}-1}}$$

[1] G. N. Lewis and M. Randall, Thermodynamics, (McGraw-Hill), 1923, pp. 164-5.

[2] P. W. Bridgman, A Condensed Collection of Thermodynamic Formulas (Harvard Univ. Press), 1925.

and

$$\frac{(\mathrm{d}W)_{①}}{(\partial m_1)_{①}} \equiv \frac{\mathrm{d}W}{\mathrm{d}m_1}, \; W = \mathrm{f}\,(\mathrm{S}),\; m_1 = \mathrm{S};\; \theta, p, m_2, \cdots, m_{n-1}$$

being constants.

$$= - p\left(\frac{\partial v}{\partial m_1}\right)_{\theta,\, p,\, m_2,\, \ldots,\, m_{n-1}}$$

The fact that in the unit mass systems we have one less variable than in the corresponding variable mass systems simplifies the tables. For example, if in the binary variable mass table, for which all the thermodynamic relations of the first derivatives are expressed in groups 1 to 92 inclusive, we replace $\mathbf{m}_1$ by m_1, and $\mathbf{m}_2$ by m_2 together with the above substitutions we have all the thermodynamic relations of the first derivatives for a ternary system of constant mass.

99. Experimental determination of the standard derivatives:

$$\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\, \mathbf{m}_1,\, \ldots,\, \mathbf{m}_n}, \quad \left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\, \mathbf{m}_1,\, \ldots,\, \mathbf{m}_n},$$

and

$$\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\, p,\, \mathbf{m}_1,\, \ldots,\, \mathbf{m}_{k-1},\, \mathbf{m}_{k+1},\, \ldots,\, \mathbf{m}_n},$$

where $k = 1, \cdots, n$, may be obtained from the characteristic equation of the system, *i.e.* from the relation between $\theta, p, \mathbf{m}_1, \cdots, \mathbf{m}_n, \mathbf{v}$. For this relationship we may measure volume as a function of θ, p, $\mathbf{m}_1, \cdots, \mathbf{m}_n$ or measure the density of the system as a function of the temperature, pressure and mass fractions of the components (see Table A, §96).

c_p will be completely determined if, in addition to knowing the characteristic equation, c_p is given as a known function of $\theta, m_1, \cdots, m_{n-1}$ at some one pressure, *e.g.* at atmospheric pressure. The effect of pressure on c_p is obtained from the characteristic equation since c_p must be such a function of $\theta, p, m_1, \cdots, m_{n-1}$ that

$$\left(\frac{\partial\, c_p}{\partial p}\right)_{\theta, m_1,\, \ldots,\, m_{n-1}} = - \theta \left[\frac{\partial^2 v}{\partial \theta^2}\right]_{p,\, m_1,\, \ldots,\, m_{n-1}}$$

Furthermore, we are required to know

$$\mu_{\mathrm{k}} = \left(\frac{\partial \zeta}{\partial \mathrm{m_k}}\right)_{\theta,\, p,\, \mathrm{m_1},\, \ldots,\, \mathrm{m_{k-1}},\, \mathrm{m_{k+1}},\, \ldots,\, \mathrm{m_n}}$$

and

$$l_{m_{\mathrm{k}}} = \left(\frac{\partial \mathbf{n}}{\partial \mathrm{m_k}}\right)_{\theta,\, p,\, \mathrm{m_1},\, \ldots,\, \mathrm{m_{k-1}},\, \mathrm{m_{k+1}},\, \ldots,\, \mathrm{m_n}}$$

where $\mathrm{k} = 1, \cdots, \mathrm{n}$. These quantities will be completely determined if, in addition to knowing the characteristic equation and c_p, μ_{k} and $l_{m_{\mathrm{k}}}$ can be expressed as known functions of the component masses at some one temperature and pressure, or if $\mu_{\mathrm{i}} - \mu_{\mathrm{n}}$, $l_{m_{\mathrm{i}}} - l_{m_{\mathrm{n}}}$, $\mathrm{i} = 1, \cdots, \mathrm{n} - 1$ are given as known functions of the mass fractions at some one temperature and pressure, *e.g.* at 20°C and one atmosphere pressure, since from §49,

$$\left(\frac{\partial \mu_{\mathrm{k}}}{\partial \theta}\right)_{p,\, \mathrm{m_1},\, \ldots,\, \mathrm{m_n}} = -\frac{l_{m_{\mathrm{k}}}}{\theta};$$

from §47,

$$\left(\frac{\partial \mu_{\mathrm{k}}}{\partial p}\right)_{\theta,\, \mathrm{m_1},\, \ldots,\, \mathrm{m_n}} = \left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\, p,\, \mathrm{m_1},\, \ldots,\, \mathrm{m_{k-1}},\, \mathrm{m_{k+1}},\, \ldots,\, \mathrm{m_n}};$$

from §43,

$$\left[\frac{\partial}{\partial \theta}\left(\frac{l_{m_{\mathrm{k}}}}{\theta}\right)\right]_{p,\, \mathrm{m_1},\, \ldots,\, \mathrm{m_n}} = \left[\frac{\partial}{\partial \mathrm{m_k}}\left(\frac{c_p}{\theta}\right)\right]_{\theta,\, p,\, \mathrm{m_1},\, \ldots,\, \mathrm{m_{k-1}},\, \mathrm{m_{k+1}},\, \ldots,\, \mathrm{m_n}};$$

and from §43,

$$\left(\frac{\partial l_{m_{\mathrm{k}}}}{\partial p}\right)_{\theta,\, \mathrm{m_1},\, \ldots,\, \mathrm{m_n}} = \left[\frac{\partial}{\partial \mathrm{m_k}}\left(-\theta\frac{\partial \mathrm{v}}{\partial \theta}\right)\right]_{\theta,\, p,\, \mathrm{m_1},\, \ldots,\, \mathrm{m_{k-1}},\, \mathrm{m_{k+1}},\, \ldots,\, \mathrm{m_n}}.$$

The most widely applicable method for obtaining μ_{k} is the freezing point method (see §100). This method, and the others listed below, for determining μ_{k} presuppose a knowledge of the thermodynamic relations between phases.

Gibbs[1] has shown that μ'_{k}, the thermodynamic potential of

[1] J. Willard Gibbs, Collected Works, vol. 1, p. 65.

constituent k in the solid state is equal to μ_k, the thermodynamic potential of constituent k in the solution, when the solid phase of k is in equilibrium with the solution.

Thus, knowing the μ'_k of the solid phase k, we can obtain the μ_k of component k in the solution along the equilibrium line of solid k and solution.

Other methods have been found useful in special systems. For example electromotive force measurements on reversible cells (see §101), can be used in binary systems to obtain μ_2 over the range of applicability provided we know the value at some one point in this range. Again osmotic pressure determinations have been used with success in special cases and may become more widely useful in the future, for Townend[1] has extended the applicability of this method to aqueous solutions of electrolytes and solutions in organic liquids, his methods being particularly applicable to dilute solutions.

l_{m_k} may be obtained directly where reversible cells can be used by measuring the temperature coefficient of the electromotive force. However, in general, l_{m_k} can not be obtained directly.

By measuring the heats of mixing in a constant volume calorimeter under non-equilibrium conditions we can get ε as a function of the component masses, further ε must be measured over the same temperature interval in which the known values of $\mu_1, \cdots, \mu_n$ lie, otherwise we must obtain ε as a function of temperature and the component masses. Then l_{m_k} can be obtained as a function of the mass fractions at the temperature and pressure at which $\dfrac{\partial \varepsilon}{\partial m_k}$ and μ_k were obtained,

$$l_{m_k} = \left(\frac{\partial \varepsilon}{\partial m_k}\right)_{\theta,\, v,\, m_1,\, \ldots,\, m_{k-1},\, m_{k+1},\, \ldots,\, m_n} - \mu_k - \theta \frac{\left(\dfrac{\partial v}{\partial \theta}\right)\left(\dfrac{\partial v}{\partial m_k}\right)}{\left(\dfrac{\partial v}{\partial p}\right)}$$

Again by measuring the heats of mixing in a constant pressure calorimeter under non-equilibrium conditions we can get χ as a

[1] R. V. Townend, J. Am. Chem. Soc., vol. 50, 1928, p. 2958.

function of the component masses. This is sufficient provided

$$\left(\frac{\partial \chi}{\partial \mathrm{m}_1}\right)_{\theta,\, p,\, \mathrm{m}_2,\, \ldots,\, \mathrm{m}_n}$$

is given for the same temperature as μ_1 was obtained where 1 = 1, · · ·, n, otherwise we must obtain χ as a function of temperature and the component masses. Then l_{m_k} can be obtained as a function of the mass fractions at the temperature and pressure at which $\frac{\partial \chi}{\partial \mathrm{m}_k}$ and μ_k were obtained,

$$l_{m_k} = \left(\frac{\partial \chi}{\partial \mathrm{m}_k}\right)_{\theta,\, p,\, \mathrm{m}_1,\, \ldots,\, \mathrm{m}_{k-1},\, \mathrm{m}_{k+1},\, \ldots,\, \mathrm{m}_n} - \mu_k.$$

100. Fundamental equation for a binary system of variable mass: "Fundamental equation" is used in the same sense as Gibbs (Collected Works, vol. 1, p. 88) defined it. For any homogeneous mass whatever, considered in general as having n independently variable components, to which the subscript numerals refer (but not excluding the case in which n = 1 and the composition of the body is invariable), there are relations between certain of the thermodynamic quantities from which, if the relations are known explicitly, with the aid only of *general* principles and relations, we may deduce all the relations subsisting for such a mass between all the thermodynamic quantities for the homogeneous mass.

Thus if we can evaluate explicitly, for a homogeneous system or phase, the functional relationship between the quantities

$$\varepsilon, \eta, \mathrm{v}, \mathrm{m}_1, \cdots, \mathrm{m}_n$$

or

$$\zeta, \theta, p, \mathrm{m}_1, \cdots, \mathrm{m}_n$$

or

$$\chi, p, \eta, \mathrm{m}_1, \cdots, \mathrm{m}_n$$

or

$$\psi, \theta, \mathrm{v}, \mathrm{m}_1, \cdots, \mathrm{m}_n$$

or

$$\theta, p, \mu_1, \cdots, \mu_n$$

we have a fundamental equation for the phase in question.

As an example we shall indicate how a fundamental equation may be obtained over all or part of a region of a binary system of variable mass.

The method illustrated will be one which is readily capable of being extended to systems containing more than two components.

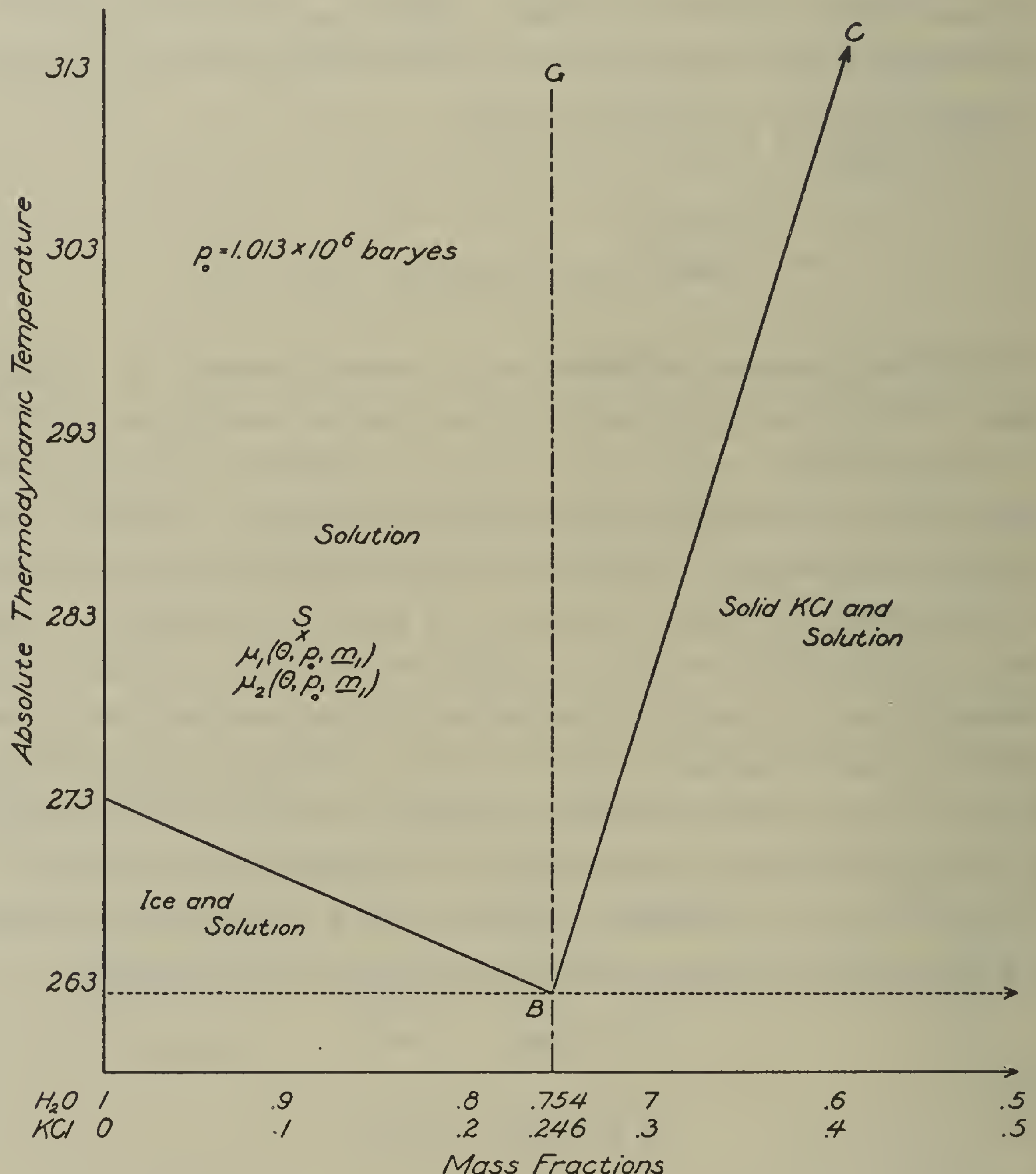

Diagram 4

The particular system chosen is the system, or phase, potassium chloride-water. Since we are not able to obtain a fundamental equation from a consideration of this single phase we must also know the thermodynamic properties of potassium chloride and of water and use the relations existing between phases in equilibrium with each other.

The density of KCl-H_2O solutions must be measured as a function of the temperature, pressure, and concentration of KCl. This gives us the characteristic equation for the phase KCl-H_2O.

Next calorimetric measurements must be made to obtain the heat capacity per unit mass at constant pressure, c_p, as a function of the temperature and concentration of KCl at some one pressure, let us say atmospheric pressure.

Let

$$\mu_{\text{ice}} = \zeta_{\text{ice}} = \mu_1'$$

$$\mu_{\text{H}_2\text{O in solution}} = \mu_1$$

$$\mu_{\text{KCl}} = \zeta_{\text{KCl}} = \mu_2'$$

$$\mu_{\text{KCl in solution}} = \mu_2$$

Along AB of diagram 4[1], which is the equilibrium line between Ice and KCl-H_2O at p = one atmosphere = 1.013×10^6 baryes,

$$\mu_1' = \mu_1.$$

But

$$\mu_1' = \zeta_{\text{ice}} = \epsilon + pv - \theta\eta$$

Now

$$\epsilon(\theta, p) - \epsilon(\theta_0, p_0) = \int_{\theta_0,\, p_0}^{\theta,\, p} \left(c_p - p\frac{\partial v}{\partial \theta}\right) d\theta - \left(\theta\frac{\partial v}{\partial \theta} + p\frac{\partial v}{\partial p}\right) dp$$

where $\epsilon(\theta_0, p_0) = 0$ by definition[2] and

$$\eta(\theta, p) - \eta(0, p) = \int_{\theta_0 = 0,\, p}^{\theta,\, p} \frac{c_p}{\theta} d\theta$$

where $\eta(0, p) = 0$ by definition.[2]

Hence we know μ_1' and therefore μ_1 along the line AB. Similarly we know μ_2' and therefore μ_2 along the line BC.

Now in order to determine μ_1 at any point S in the region ABG of diagram 4, we must know

$$l_{m_1} = -\theta\left(\frac{\partial \mu_1}{\partial \theta}\right)_{p,\, m_1,\, \dots,\, m_n}.$$

[1] Coordinates (0,273) should be denoted by A.

[2] See § 93 for justification of these definitions.

As we found in §99 it is sufficient to know l_{m_1} as a function of the concentration at some one temperature and pressure.

To do this we shall measure the heats of mixing under non-equilibrium conditions in a constant pressure calorimeter to obtain χ as a function of the temperatuie range CH_1,[1] *i.e.* the temperature range in which the known values of μ_1 and μ_2 lie, and the component masses $\mathbf{m}_1$ and $\mathbf{m}_2$ at atmospheric pressure.

Thus we get

$$l_{m_1} = \left(\frac{\partial \chi}{\partial \mathbf{m}_1}\right)_{\theta,\, p,\, \mathbf{m}_2} - \mu_1$$

$$l_{m_2} = \left(\frac{\partial \chi}{\partial \mathbf{m}_2}\right)_{\theta,\, p,\, \mathbf{m}_1} - \mu_2$$

Hence we have l_{m_1} and l_{m_2} as known functions of θ, p, m_1 since, from §99,

$$\frac{\partial}{\partial \theta}\left(\frac{l_{m_1}}{\theta}\right) = \frac{\partial}{\partial \mathbf{m}_1}\left(\frac{c_p}{\theta}\right) \text{and} \frac{\partial\, l_{m_1}}{\partial p} = \frac{\partial}{\partial \mathbf{m}_1}\left(-\,\theta\,\frac{\partial \mathbf{v}}{\partial \theta}\right)$$

Thus we now have μ_1 in the region GBA and μ_2 in the region GBC. We still have to determine μ_2 in GBA and μ_1 in GBC.

At B, the triple point, the three phases ice, KCl, and solution are in equilibrium, hence

$$\mu_1'\,(262.6°, p) = \mu_2'\,(262.6°, p) = \mu_1\,(262.6°, p, 0.754)$$

where $p = 1.013 \times 10^6$ baryes.

[1] If we know χ as a function of θ, $\mathbf{m}_1$, $\mathbf{m}_2$ then we have directly

$$\chi = \zeta + \theta\,\mathbf{n}$$

$$\left(\frac{\partial \chi}{\partial \mathbf{m}_1}\right)_{\theta,\, p,\, \mathbf{m}_2} = \left(\frac{\partial \zeta}{\partial \mathbf{m}_1}\right)_{\theta,\, p,\, \mathbf{m}_2} + \theta\left(\frac{\partial \mathbf{n}}{\partial \mathbf{m}_1}\right)_{\theta,\, p,\, \mathbf{m}_2}$$

$$= \mu_1 - \theta\left(\frac{\partial \mu_1}{\partial \theta}\right)_{p,\, \mathbf{m}_1,\, \mathbf{m}_2}$$

$$= -\,\theta^2\left[\frac{\partial}{\partial \theta}\left(\frac{\mu_1}{\theta}\right)\right]_{p,\, \mathbf{m}_1,\, \mathbf{m}_2}$$

Thus

$$\frac{\mu_1\,(\theta_2, p_1, m_1)}{\theta_2} - \frac{\mu_1\,(\theta_1, p_1, m_1)}{\theta_1} = -\int_{\theta_1,\, p_1,\, \mathbf{m}_1,\, \mathbf{m}_2}^{\theta_2,\, p_1,\, \mathbf{m}_1,\, \mathbf{m}_2} \frac{1}{\theta^2}\left(\frac{\partial \chi}{\partial \mathbf{m}_1}\right)_{\theta,\, p,\, \mathbf{m}_2} d\theta$$

But we know μ_1 as a function of temperature,

$$\left(\frac{\partial \mu_1}{\partial \theta}\right)_{p,\, m_1} = -\frac{l_{m_1}}{\theta}$$

And from §50 we have $m_1\, d\mu_1 + m_2\, d\mu_2 = 0$ where temperature and pressure are constant.

Therefore

$$\mu_2(\theta, p, m_1) - \mu_2(\theta, p, m_{1_0}) = \int_{\theta,\, p,\, m_{1_0}}^{\theta,\, p,\, m_1} \frac{m_1}{m_2}\, d\mu_1$$

which gives us μ_2 in the region GBA. Similarly we get μ_1 in the region GBC.

Hence we now have v, c_p, μ_1, μ_2, l_{m_1} and l_{m_2} as known functions of θ, p, m_1 which is all we need to know in order to formulate a fundamental equation for a binary system.

101. Electromotive force measurements: If our binary solution is an electrolyte, which is the case for our KCl-H_2O[1] solution except at infinite dilution, it is sometimes possible, if we know the value of μ_2 at one point in the region in which this method is applicable, to obtain μ_2 and l_{m_2} rather readily by experiment. Accurate measurements of electromotive force, however, become increasingly difficult at high dilution and therefore of increasingly doubtful value.

We shall set up a double cell in which the process is reversible, *i.e.* can be made to go forward and backward through the same series of states or path. Furthermore our cell is one in which the chemical reaction or "transference" is known.

The electrolyte in the cells is a potassium chloride solution, KCl(aq). In cell I the concentration of potassium chloride is m_2 and in cell II is m_{2_0}.

$$\mathrm{Ag + KCl\,(aq)} \xrightarrow{\mathrm{I}} \mathrm{K + Ag\,Cl + aq} \xrightarrow{\mathrm{II}} \mathrm{Ag + KCl\,(aq)}$$

The net result is to transfer a certain amount, $\mathbf{m}_2$ grams, of KCl from the solution I to solution II.

[1] Duncan A. MacInnes and Karr Parker, J. A. C. S., vol. 37, 1915, pp. 1449-1455.

We shall consider this double cell as a whole and to simplify nomenclature write a prime, as ζ' where we mean ζ for the combined cells (ζ of I + ζ of II).

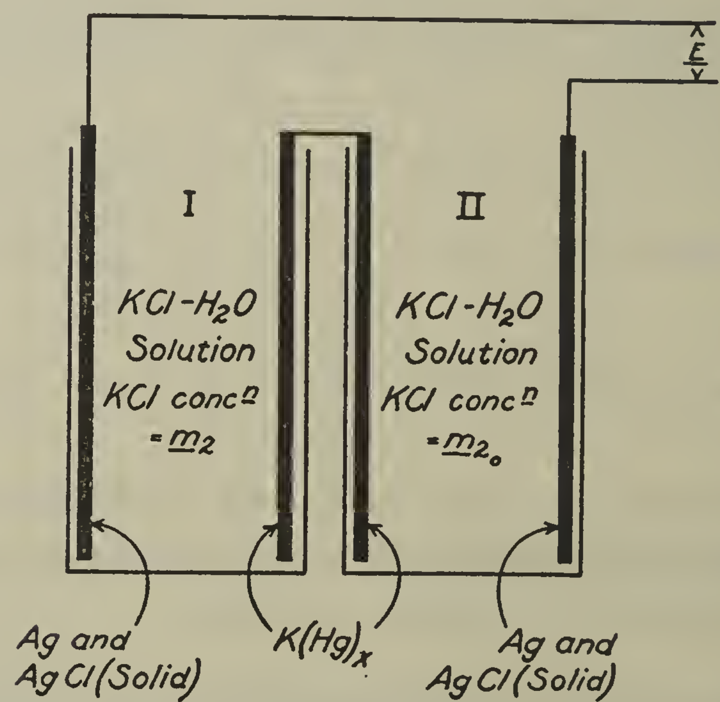

DIAGRAM 5

Now since the process in the cell is a reversible one[1] taking place at constant temperature and pressure we can write

$$Q' = \int_{n'_0}^{n'} \theta \, dn = \theta (n' - n_0')$$

where n'_0 and n' denote the total entropy of the combined cells at the initial and final states respectively.

The mechanical and electrical work W' and W_e' respectively

$$W' + W_e' = - \int_{v'_0,\, q = 0}^{v',\, q = q} p \, dv' + E \, dq$$

$$= - p (v' - v_0') - E q$$

where E denotes the electromotive force and q the quantity of electricity that flows.

The total energy change is then

$$\varepsilon' - \varepsilon_0' = \theta (n' - n_0') - p (v' - v_0') - E q$$

[1] See §92, for definition of "reversible process."

or

$$\zeta' - \zeta_0' = -E\,\mathrm{q}$$

But

$$\zeta' - \zeta_0' = \mathbf{m}_2\,\mu_2 - \mathbf{m}_2\,\mu_{2_0}$$

where $\mathbf{m}_2$ is the mass of KCl transferred from cell I to cell II, the water remaining constant.

Hence

$$\mu_2 - \mu_{2_0} = -\frac{E\,\mathrm{q}}{\mathbf{m}_2}$$

which gives us $\mu_2\ (t,\ p,\ m_1)$ if we know $\mu_{2_0}\ (t,\ p,\ m_{1_0})$.

From the temperature coefficient of the electromotive force we obtain l_{m_2} since

$$\left(\frac{\partial\mu_2}{\partial\theta}\right)_{p,\ \mathrm{m}_1,\ \mathrm{m}_2} = -\frac{l_{m_2}}{\theta}.$$

This method, however, lacks the general applicability of the previous method discussed.

102. The second derivatives (Table II): The second derivatives may be divided into two groups.

Any second derivative formed from the first n + 8 variable uantities is of the type

$$\left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\ \mathrm{m}_1,\ \ldots,\ \mathrm{m}_n}\right] = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial\theta}\right)_{p,\ \mathrm{m}_1,\ \ldots,\ \mathrm{m}_n}\right]_{\theta,\ \mathrm{m}_1,\ \ldots,\ \mathrm{m}_n}$$

Any second derivative which includes the work and heat is of the type

$$\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{dW}}{\mathrm{d}p}\right)_{\theta,\ \mathrm{m}_1,\ \ldots,\ \mathrm{m}_n}\right]_{p,\ \mathrm{m}_1,\ \ldots,\ \mathrm{m}_n} \neq \left[\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\mathrm{dW}}{\mathrm{d}\theta}\right)_{p,\ \mathrm{m}_1,\ \ldots,\ \mathrm{m}_n}\right]_{\theta,\ \mathrm{m}_1,\ \ldots,\ \mathrm{m}_n}$$

The number of combinations of second derivatives is so great that it can not be reduced to a reasonable number, as could the number of combinations of first derivatives.

All that has been attempted here is to tabulate a number of the standard second derivatives so that the relations thought to be of most use may be found readily by ordinary formal differentiation and algebraic elimination.

It can easily be shown (see §99) that there are $4 + 3n + 3n^2$ standard second derivatives,

$$\frac{\partial^2 \mathbf{v}}{\partial \theta^2};\ \frac{\partial^2 \mathbf{v}}{\partial \theta\, \partial p};\ \frac{\partial^2 \mathbf{v}}{\partial \theta\, \partial \mathbf{m}_k};\ \frac{\partial^2 \mathbf{v}}{\partial p^2};\ \frac{\partial^2 \mathbf{v}}{\partial p\, \partial \mathbf{m}_k};\ \frac{\partial^2 \mathbf{v}}{\partial \mathbf{m}_h\, \partial \mathbf{m}_k};\ (\mathbf{m}_1 + \cdots$$

$$+ \mathbf{m}_n)\frac{\partial c_p}{\partial \theta};\ \frac{\partial}{\partial \mathbf{m}_k}\Big[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\Big];\ \frac{\partial \mu_k}{\partial \mathbf{m}_h};\ \text{and}\ \frac{\partial\, l_{m_k}}{\partial \mathbf{m}_h};$$

where $(h, k) = 1, \ldots, n$.

As an example we shall evaluate

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{n}}{\partial \theta}\right)_{p,\,\mathbf{m}_1,\,\ldots,\,\mathbf{m}_n}\right]_{\theta,\,\mathbf{m}_1,\,\ldots,\,\mathbf{m}_n}$$

by means of the tables.

$$\left(\frac{\partial \mathbf{n}}{\partial \theta}\right)_{p,\,\mathbf{m}_1,\,\ldots,\,\mathbf{m}_n} = (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}$$

from Table I Group 2.

$$\begin{aligned}
\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{n}}{\partial \theta}\right)_{p,\,\mathbf{m}_1,\,\ldots,\,\mathbf{m}_n}\right]_{\theta,\,\mathbf{m}_1,\,\ldots,\,\mathbf{m}_n}
&= \left[\frac{\partial}{\partial p}\left\{(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\right\}\right]_{\theta,\,\mathbf{m}_1,\,\ldots,\,\mathbf{m}_n} \\
&= \frac{1}{\theta}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\left(\frac{\partial c_p}{\partial p}\right)_{\theta,\,\mathbf{m}_1,\,\ldots,\,\mathbf{m}_n} \\
&= -\left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\,\mathbf{m}_1,\,\ldots,\,\mathbf{m}_n}
\end{aligned}$$

from Table II Group 1.

In dealing with the second derivatives of **Q** and **W** care must be used since, in general, if the order of differentiation is changed the value of the second derivative is changed, for **Q** and **W** are not functions of the state of the system but of the path traversed in reaching that state.

For example

$$\left[\frac{d}{dp}\left(\frac{dW}{d\theta}\right)_{p,\, m_1,\, \ldots,\, m_n}\right]_{\theta,\, m_1,\, \ldots,\, m_n}$$

$$= -\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_1,\, \ldots,\, m_n} - p\left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_1,\, \ldots,\, m_n}\right]_{\theta,\, m_1,\, \ldots,\, m_n};$$

$$\left[\frac{d}{d\theta}\left(\frac{dW}{dp}\right)_{\theta,\, m_1,\, \ldots,\, m_n}\right]_{p,\, m_1,\, \ldots,\, m_n}$$

$$= -p\left[\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_1,\, \ldots,\, m_n}\right]_{p,\, m_1,\, \ldots,\, m_n}.$$

Table II has not been completed as far as Table I because the expressions become very clumsy and furthermore the extension of this table can be made rather readily by formal differentiation which gives derivatives that can be evaluated by Tables I and II. For example suppose we wish second derivatives where m_g, v, ε, n are held constant. Let us choose one of the derivatives which may be involved, namely,

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial m_k}\right)_{\theta,\, p,\, m_i}\right]_{m_g,\, v,\, e,\, n}$$

Now by formal differentiation we have

$$\frac{\partial^2 v}{\partial p\, \partial m_k} = \left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial m_k}\right)_{\theta,\, p,\, m_i}\right]_{\theta,\, m_1,\, \ldots,\, m_n} +$$
$$\left[\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial m_k}\right)_{\theta,\, p,\, m_i}\right]_{p,\, m_1,\, \ldots,\, m_n}\left(\frac{\partial \theta}{\partial p}\right)_{m_g,\, v,\, e,\, n} +$$
$$\left(\frac{\partial^2 v}{\partial m_k^2}\right)_{\theta,\, p,\, m_i}\left(\frac{\partial m_k}{\partial p}\right)_{m_g,\, v,\, e,\, n} +$$
$$\left[\frac{\partial}{\partial m_h}\left(\frac{\partial v}{\partial m_k}\right)_{\theta,\, p,\, m_i}\right]_{\theta,\, p,\, m_j}\left(\frac{\partial m_h}{\partial p}\right)_{m_g,\, v,\, e,\, n}$$

The values of the first and second derivatives on the right hand side of the equation are given in group 73 of Table I, and in groups 1, 2, and 9 of Table II respectively.

PART II

KEY TO TABLES

The numeral denotes the group number, the letters following are the variables held constant in the group.

All the component masses constant. Thus the groups in this section include all the thermodynamic relations existing for one component unit mass systems (constant total mass).

1. θ, $\mathbf{m}_1 \ldots, \mathbf{m}_n$	4. $\mathbf{m}_1, \ldots, \mathbf{m}_n$, ε	7. $\mathbf{m}_1, \ldots, \mathbf{m}_n$, χ
2. p, $\mathbf{m}_1, \ldots, \mathbf{m}_n$	5. $\mathbf{m}_1, \ldots, \mathbf{m}_n$, $\mathfrak{n}$	8. $\mathbf{m}_1, \ldots, \mathbf{m}_n$, ψ
3. $\mathbf{m}_1, \ldots, \mathbf{m}_n$, v	6. $\mathbf{m}_1, \ldots, \mathbf{m}_n$, ζ	

All the component masses but one constant. Thus the groups in this section, together with the above groups, include all the thermodynamic relations for one component variable mass systems or two component unit mass systems (total mass constant).

9. θ, p, $\mathbf{m}_i$*	19. p, $\mathbf{m}_i$, ζ	29. $\mathbf{m}_i$, ε, χ
10. θ, $\mathbf{m}_i$, v	20. p, $\mathbf{m}_i$, χ	30. $\mathbf{m}_i$, ε, ψ
11. θ, $\mathbf{m}_i$, ε	21. p, $\mathbf{m}_i$, ψ	31. $\mathbf{m}_i$, $\mathfrak{n}$, ζ
12. θ, $\mathbf{m}_i$, $\mathfrak{n}$	22. $\mathbf{m}_i$, v, ε	32. $\mathbf{m}_i$, $\mathfrak{n}$, χ
13. θ, $\mathbf{m}_i$, ζ	23. $\mathbf{m}_i$, v, $\mathfrak{n}$	33. $\mathbf{m}_i$, $\mathfrak{n}$, ψ
14. θ, $\mathbf{m}_i$, χ	24. $\mathbf{m}_i$, v, ζ	34. $\mathbf{m}_i$, ζ, χ
15. θ, $\mathbf{m}_i$, ψ	25. $\mathbf{m}_i$, v, χ	35. $\mathbf{m}_i$, ζ, ψ
16. p, $\mathbf{m}_i$, v	26. $\mathbf{m}_i$, v, ψ	36. $\mathbf{m}_i$, χ, ψ
17. p, $\mathbf{m}_i$, ε	27. $\mathbf{m}_i$, ε, $\mathfrak{n}$	
18. p, $\mathbf{m}_i$, $\mathfrak{n}$	28. $\mathbf{m}_i$, ε, ζ	

All the component masses but two constant. Thus the groups in this section, together with the above, include all thermodynamic relations existing for two component variable mass systems or three component unit mass systems (constant total mass).

37. θ, p, $\mathbf{m}_g$, v*	40. θ, p, $\mathbf{m}_g$, ζ	43. θ, $\mathbf{m}_g$, v, ε
38. θ, p, $\mathbf{m}_g$, ε	41. θ, p, $\mathbf{m}_g$, χ	44. θ, $\mathbf{m}_g$, v, $\mathfrak{n}$
39. θ, p, $\mathbf{m}_g$, $\mathfrak{n}$	42. θ, p, $\mathbf{m}_g$, ψ	45. θ, $\mathbf{m}_g$, v, ζ

* $\mathbf{m}_i$ denotes all the component masses except $\mathbf{m}_k$; $\mathbf{m}_g$ all except $\mathbf{m}_k$ and $\mathbf{m}_h$.

46. θ, m_g, v, χ
47. θ, m_g, v, ψ
48. $\theta, m_g, \varepsilon, n$
49. $\theta, m_g, \varepsilon, \zeta$
50. $\theta, m_g, \varepsilon, \chi$
51. $\theta, m_g, \varepsilon, \psi$
52. θ, m_g, n, ζ
53. θ, m_g, n, χ
54. θ, m_g, n, ψ
55. θ, m_g, ζ, χ
56. θ, m_g, ζ, ψ
57. θ, m_g, χ, ψ
58. p, m_g, v, ε
59. p, m_g, v, n
60. p, m_g, v, ζ
61. p, m_g, v, χ
62. p, m_g, v, ψ
63. p, m_g, ε, n
64. $p, m_g, \varepsilon, \zeta$
65. $p, m_g, \varepsilon, \chi$
66. $p, m_g, \varepsilon, \psi$
67. p, m_g, n, ζ
68. p, m_g, n, χ
69. p, m_g, n, ψ
70. p, m_g, ζ, χ
71. p, m_g, ζ, ψ
72. p, m_g, χ, ψ
73. m_g, v, ε, n
74. $m_g, v, \varepsilon, \zeta$
75. $m_g, v, \varepsilon, \chi$
76. $m_g, v, \varepsilon, \psi$
77. m_g, v, n, ζ
78. m_g, v, n, χ
79. m_g, v, n, ψ
80. m_g, v, ζ, χ
81. m_g, v, ζ, ψ
82. m_g, v, χ, ψ
83. $m_g, \varepsilon, n, \zeta$
84. $m_g, \varepsilon, n, \chi$
85. $m_g, \varepsilon, n, \psi$
86. $m_g, \varepsilon, \zeta, \chi$
87. $m_g, \varepsilon, \zeta, \psi$
88. $m_g, \varepsilon, \chi, \psi$
89. m_g, n, ζ, χ
90. m_g, n, ζ, ψ
91. m_g, n, χ, ψ
92. m_g, ζ, χ, ψ

All the component masses but three constant. Thus the groups in this section, together with the above, include all the thermodynamic relations existing for the three component variable mass systems or four component unit mass systems (total mass constant). This also includes all the relations thought to be of practical value for systems of more than four components.

93. $\theta, p, m_b, v, \varepsilon$*
94. θ, p, m_b, v, n
95. θ, p, m_b, v, ζ
96. θ, p, m_b, v, χ
97. θ, p, m_b, v, ψ
98. $\theta, p, m_b, \varepsilon, n$
99. $\theta, p, m_b, \varepsilon, \zeta$
100. $\theta, p, m_b, \varepsilon, \chi$
101. $\theta, p, m_b, \varepsilon, \psi$
102. θ, p, m_b, n, ζ
103. θ, p, m_b, n, χ
104. θ, p, m_b, n, ψ
105. $\theta, p, m_b, \zeta, \chi$
106. $\theta, p, m_b, \zeta, \psi$
107. $\theta, p, m_b, \chi, \psi$
108. $\theta, m_b, v, \varepsilon, n$
109. $\theta, m_b, v, \varepsilon, \zeta$
110. $\theta, m_b, v, \varepsilon, \chi$
111. $\theta, m_b, v, \varepsilon, \psi$
112. θ, m_b, v, n, ζ
113. θ, m_b, v, n, χ
114. θ, m_b, v, n, ψ
115. $\theta, m_b, v, \zeta, \chi$
116. $\theta, m_b, v, \zeta, \psi$
117. $\theta, m_b, v, \chi, \psi$
118. $\theta, m_b, \varepsilon, n, \zeta$
119. $\theta, m_b, \varepsilon, n, \chi$
120. $\theta, m_b, \varepsilon, n, \psi$
121. $\theta, m_b, \varepsilon, \zeta, \chi$
122. $\theta, m_b, \varepsilon, \zeta, \psi$
123. $\theta, m_b, \varepsilon, \chi, \psi$
124. $\theta, m_b, n, \zeta, \chi$
125. $\theta, m_b, n, \zeta, \psi$
126. $\theta, m_b, n, \chi, \psi$
127. $\theta, m_b, \zeta, \chi, \psi$
128. $p, m_b, v, \varepsilon, n$
129. $p, m_b, v, \varepsilon, \zeta$
130. $p, m_b, v, \varepsilon, \chi$
131. $p, m_b, v, \varepsilon, \psi$

* m_b denotes all the component masses except m_h, m_k, m_y.

132. p, m_b, v, n, ζ
133. p, m_b, v, n, χ
134. p, m_b, v, n, ψ
135. p, m_b, v, ζ, χ
136. p, m_b, v, ζ, ψ
137. p, m_b, v, χ, ψ
138. p, m_b, ε, n, ζ
139. p, m_b, ε, n, χ
140. p, m_b, ε, n, ψ
141. p, m_b, ε, ζ, χ
142. p, m_b, ε, ζ, ψ
143. p, m_b, ε, χ, ψ
144. p, m_b, n, ζ, χ
145. p, m_b, n, ζ, ψ
146. p, m_b, n, χ, ψ
147. p, m_b, ζ, χ, ψ
148. m_b, v, ε, n, ζ
149. m_b, v, ε, n, χ
150. m_b, v, ε, n, ψ
151. m_b, v, ε, ζ, χ
152. m_b, v, ε, ζ, ψ
153. m_b, v, ε, χ, ψ
154. m_b, v, n, ζ, χ
155. m_b, v, n, ζ, ψ
156. m_b, v, n, χ, ψ
157. m_b, v, ζ, χ, ψ
158. m_b, ε, n, ζ, χ
159. m_b, ε, n, ζ, ψ
160. m_b, ε, n, χ, ψ
161. m_b, ε, ζ, χ, ψ
162. m_b, n, ζ, χ, ψ

TABLE I

First Derivatives

Group 1

$\theta, \mathbf{m}_1, \cdots, \mathbf{m}_n$ constant.

$$(\partial p) = -1$$

$$(\partial \mathbf{v}) = -\frac{\partial \mathbf{v}}{\partial p}.$$

$$(\partial \boldsymbol{\varepsilon}) = p\frac{\partial \mathbf{v}}{\partial p} + \theta\frac{\partial \mathbf{v}}{\partial \theta}$$

$$(\partial \mathbf{n}) = \frac{\partial \mathbf{v}}{\partial \theta}$$

$$(\partial \zeta) = -\mathbf{v}$$

$$(\partial \chi) = -\left[\mathbf{v} - \theta\frac{\partial \mathbf{v}}{\partial \theta}\right]$$

$$(\partial \psi) = p\frac{\partial \mathbf{v}}{\partial p}$$

$$(\mathrm{d}\mathbf{W}) = p\frac{\partial \mathbf{v}}{\partial p}$$

$$(\mathrm{d}\mathbf{Q}) = \theta\frac{\partial \mathbf{v}}{\partial \theta}$$

Group 2

$p, \mathbf{m}_1, \cdots, \mathbf{m}_n$ constant.

$$(\partial \theta) = 1$$

$$(\partial \mathbf{v}) = \frac{\partial \mathbf{v}}{\partial \theta}$$

$$(\partial \boldsymbol{\varepsilon}) = (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p - p\frac{\partial \mathbf{v}}{\partial \theta}$$

Group 2 (Con.)

$$(\partial \mathbf{n}) = \frac{1}{\theta}(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p$$

$$(\partial \zeta) = -\,\mathbf{n}$$

$$(\partial \chi) = (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p$$

$$(\partial \psi) = -\,p\frac{\partial \mathrm{v}}{\partial \theta} - \mathbf{n}$$

$$(d\mathrm{W}) = -\,p\frac{\partial \mathrm{v}}{\partial \theta}$$

$$(d\mathrm{Q}) = c_p\,(\mathrm{m}_1 + \cdots + \mathrm{m}_n)$$

Group 3

$\mathrm{m}_1 \cdots, \mathrm{m}_n, \mathrm{v}$ constant.

$$(\partial \theta) = \frac{\partial \mathrm{v}}{\partial p}$$

$$(\partial p) = -\frac{\partial \mathrm{v}}{\partial \theta}$$

$$(\partial \varepsilon) = (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p\frac{\partial \mathrm{v}}{\partial p} + \theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2$$

$$(\partial \mathbf{n}) = \frac{1}{\theta}\left\{(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p\frac{\partial \mathrm{v}}{\partial p} + \theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2\right\}$$

$$(\partial \zeta) = -\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} - \mathbf{n}\frac{\partial \mathrm{v}}{\partial p}$$

$$(\partial \chi) = -\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} + (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p\frac{\partial \mathrm{v}}{\partial p} + \theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2$$

$$(\partial \psi) = -\,\mathbf{n}\frac{\partial \mathrm{v}}{\partial p}$$

$$(d\mathrm{W}) = 0$$

$$(d\mathrm{Q}) = (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p\frac{\partial \mathrm{v}}{\partial p} + \theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2$$

Group 4

$m_1, \cdots, m_n, \varepsilon$ constant.

$$(\partial\theta) = -\theta\frac{\partial v}{\partial\theta} - p\frac{\partial v}{\partial p}$$

$$(\partial p) = -(m_1 + \cdots + m_n)\,c_p + p\frac{\partial v}{\partial\theta}$$

$$(\partial v) = -(m_1 + \cdots + m_n)\,c_p\frac{\partial v}{\partial p} - \theta\left(\frac{\partial v}{\partial\theta}\right)^2$$

$$(\partial\eta) = -\frac{p}{\theta}(m_1 + \cdots + m_n)\,c_p\frac{\partial v}{\partial p} - p\left(\frac{\partial v}{\partial\theta}\right)^2$$

$$(\partial\zeta) = \eta\left(\theta\frac{\partial v}{\partial\theta} + p\frac{\partial v}{\partial p}\right) - v\left[(m_1 + \cdots + m_n)\,c_p - p\frac{\partial v}{\partial\theta}\right]$$

$$(\partial\chi) = -(m_1 + \cdots + m_n)\,c_p\left(p\frac{\partial v}{\partial p} + v\right) - p\frac{\partial v}{\partial\theta}\left(\theta\frac{\partial v}{\partial\theta} - v\right)$$

$$(\partial\psi) = \eta\left(\theta\frac{\partial v}{\partial\theta} + p\frac{\partial v}{\partial p}\right) + p\left[c_p(m_1 + \cdots + m_n)\frac{\partial v}{\partial p} + \theta\left(\frac{\partial v}{\partial\theta}\right)^2\right]$$

$$(dW) = p(m_1 + \cdots + m_n)\,c_p\frac{\partial v}{\partial p} + p\,\theta\left(\frac{\partial v}{\partial\theta}\right)^2$$

$$(dQ) = -p(m_1 + \cdots + m_n)\,c_p\frac{\partial v}{\partial p} - p\,\theta\left(\frac{\partial v}{\partial\theta}\right)^2$$

Group 5

$m_1, \cdots, m_n, \eta,$ constant.

$$(\partial\theta) = -\left(\frac{\partial v}{\partial\theta}\right)$$

$$(\partial p) = -\frac{(m_1 + \cdots + m_n)\,c_p}{\theta}$$

$$(\partial v) = -\frac{1}{\theta}(m_1 + \cdots + m_n)\,c_p\frac{\partial v}{\partial p} - \left(\frac{\partial v}{\partial\theta}\right)^2$$

$$(\partial\varepsilon) = \frac{p}{\theta}\left[(m_1 + \cdots + m_n)\,c_p\frac{\partial v}{\partial p} + \theta\left(\frac{\partial v}{\partial\theta}\right)^2\right]$$

Group 5 (Con.)

$$(\partial\zeta) = -(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\mathbf{v} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta}$$

$$(\partial\chi) = -(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\mathbf{v}$$

$$(\partial\psi) = \frac{p}{\theta}\left[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \frac{\partial \mathbf{v}}{\partial p} + \theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2\right] + \mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta}$$

$$(\mathrm{d}\mathbf{W}) = \frac{p}{\theta}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \frac{\partial \mathbf{v}}{\partial p} + p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2$$

$$(\mathrm{d}\mathbf{Q}) = 0$$

Group 6

$\mathbf{m}_1, \cdots, \mathbf{m}_n, \zeta$ constant.

$$(\partial\theta) = \mathbf{v}$$

$$(\partial p) = \mathbf{n}$$

$$(\partial \mathbf{v}) = \mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}$$

$$(\partial\varepsilon) = -\mathbf{n}\left(\theta\frac{\partial \mathbf{v}}{\partial \theta} + p\frac{\partial \mathbf{v}}{\partial p}\right) + \mathbf{v}\left[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p - p\frac{\partial \mathbf{v}}{\partial \theta}\right]$$

$$(\partial\mathbf{n}) = (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\mathbf{v} - \mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta}$$

$$(\partial\chi) = (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,\mathbf{v}\, c_p + \mathbf{n}\left(\mathbf{v} - \theta\frac{\partial \mathbf{v}}{\partial \theta}\right)$$

$$(\partial\psi) = -p\left(\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}\right) - \mathbf{v}\,\mathbf{n}$$

$$(\mathrm{d}\mathbf{W}) = -p\,\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} - p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial p}$$

$$(\mathrm{d}\mathbf{Q}) = (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \mathbf{v} - \theta\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta}$$

Group 7

$\mathbf{m}_1, \cdots, \mathbf{m}_n, \chi$ constant

$$(\partial\theta) = \mathbf{v} - \theta\frac{\partial \mathbf{v}}{\partial \theta}$$

$$(\partial p) = -(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p$$

$$(\partial \mathbf{v}) = \mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} - c_p(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{\partial \mathbf{v}}{\partial p} - \theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2$$

$$(\partial \boldsymbol{\varepsilon}) = (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right) + p\frac{\partial \mathbf{v}}{\partial \theta}\left(\theta\frac{\partial \mathbf{v}}{\partial \theta} - \mathbf{v}\right)$$

$$(\partial \mathfrak{n}) = \frac{1}{\theta}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\, \mathbf{v}$$

$$(\partial \zeta) = -(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\, \mathbf{v} - \mathfrak{n}\left(\mathbf{v} - \theta\frac{\partial \mathbf{v}}{\partial \theta}\right)$$

$$(\partial \psi) = -\mathfrak{n}\left(\mathbf{v} - \theta\frac{\partial \mathbf{v}}{\partial \theta}\right) - p\left[\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\frac{\partial \mathbf{v}}{\partial p} - \theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2\right]$$

$$(d\mathbf{W}) = -p\left[\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\frac{\partial \mathbf{v}}{\partial p} - \theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2\right]$$

$$(d\mathbf{Q}) = (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\, \mathbf{v}$$

Group 8

$\mathbf{m}_1, \cdots, \mathbf{m}_n, \psi$ constant

$$(\partial\theta) = -p\frac{\partial \mathbf{v}}{\partial p}$$

$$(\partial p) = p\frac{\partial \mathbf{v}}{\partial \theta} + \mathfrak{n}$$

$$(\partial \mathbf{v}) = \mathfrak{n}\frac{\partial \mathbf{v}}{\partial p}$$

$$(\partial \boldsymbol{\varepsilon}) = -\mathfrak{n}\left(\theta\frac{\partial \mathbf{v}}{\partial \theta} + p\frac{\partial \mathbf{v}}{\partial p}\right) - p\left[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\frac{\partial \mathbf{v}}{\partial p} + \theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2\right]$$

Group 8 (Con.)

$$(\partial \mathbf{n}) = -\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta} - p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2 - p\frac{c_p}{\theta}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{\partial \mathbf{v}}{\partial p}$$

$$(\partial \zeta) = \mathbf{n}\,\mathbf{v} + p\left(\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}\right)$$

$$(\partial \chi) = \mathbf{n}\left(\mathbf{v} - \theta\frac{\partial \mathbf{v}}{\partial \theta}\right) + p\left[\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\frac{\partial \mathbf{v}}{\partial p} - \theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2\right]$$

$$(d\mathbf{W}) = -\,p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial p}$$

$$(d\mathbf{Q}) = -\,\theta\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta} - \theta\,p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2 - p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\frac{\partial \mathbf{v}}{\partial p}.$$

Group 9

$\theta, p, \mathbf{m}_i$ constant.

$$(\partial \mathbf{m}_k) = 1$$

$$(\partial \mathbf{v}) = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\, p,\, \mathbf{m}_1,\, \ldots,\, \mathbf{m}_{k-1},\, \mathbf{m}_{k+1},\, \ldots,\, \mathbf{m}_n}$$

$$(\partial \varepsilon) = \mu_k + l_{m_k} - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial \mathbf{n}) = \frac{l_{m_k}}{\theta}$$

$$(\partial \zeta) = \mu_k$$

$$(\partial \chi) = \mu_k + l_{m_k}$$

$$(\partial \psi) = \mu_k - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(d\mathbf{W}) = -\,p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(d\mathbf{Q}) = l_{m_k}$$

Group 10

θ, $\mathbf{m}_i$, $\mathbf{v}$ constant

$$(\partial p) = -\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial \mathbf{m}_k) = \frac{\partial \mathbf{v}}{\partial p}$$

$$(\partial \boldsymbol{\varepsilon}) = (\mu_k + l_{m_k})\frac{\partial \mathbf{v}}{\partial p} + \theta\frac{\partial \mathbf{v}}{\partial \theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial \boldsymbol{\eta}) = \frac{1}{\theta}l_{m_k}\frac{\partial \mathbf{v}}{\partial p} + \frac{\partial \mathbf{v}}{\partial \theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial \boldsymbol{\zeta}) = \mu_k\frac{\partial \mathbf{v}}{\partial p} - \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial \boldsymbol{\chi}) = (\mu_k + l_{m_k})\frac{\partial \mathbf{v}}{\partial p} + \left(\theta\frac{\partial \mathbf{v}}{\partial \theta} - \mathbf{v}\right)\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial \boldsymbol{\psi}) = \mu_k\frac{\partial \mathbf{v}}{\partial p}$$

$$(d\mathbf{W}) = 0$$

$$(d\mathbf{Q}) = l_{m_k}\frac{\partial \mathbf{v}}{\partial p} + \theta\frac{\partial \mathbf{v}}{\partial \theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}.$$

Group 11

θ, $\mathbf{m}_i$, $\boldsymbol{\varepsilon}$ constant.

$$(\partial p) = -\left[\mu_k + l_{m_k} - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right]$$

$$(\partial \mathbf{m}_k) = -\left[\theta\frac{\partial \mathbf{v}}{\partial \theta} + p\frac{\partial \mathbf{v}}{\partial p}\right]$$

$$(\partial \mathbf{v}) = -\left[(\mu_k + l_{m_k})\frac{\partial \mathbf{v}}{\partial p} + \theta\frac{\partial \mathbf{v}}{\partial \theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right]$$

$$(\partial \boldsymbol{\eta}) = \left(\mu_k - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)\frac{\partial \mathbf{v}}{\partial \theta} - \frac{p}{\theta}l_{m_k}\frac{\partial \mathbf{v}}{\partial p}$$

Group 11 (Con.)

$$(\partial\zeta) = -\mu_k\left(\theta\frac{\partial \mathrm{v}}{\partial\theta} + p\frac{\partial \mathrm{v}}{\partial p}\right) + \mathrm{v}\left(p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} - \mu_k - l_{m_k}\right)$$

$$(\partial\chi) = -(\mu_k + l_{m_k})\left(p\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right) - \left(\theta\frac{\partial \mathrm{v}}{\partial\theta} - \mathrm{v}\right)p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}$$

$$(\partial\psi) = -\left(\mu_k - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right)\theta\frac{\partial \mathrm{v}}{\partial\theta} + p\,l_{m_k}\frac{\partial \mathrm{v}}{\partial p}$$

$$(\mathrm{d}W) = (\mu_k + l_{m_k})\,p\frac{\partial \mathrm{v}}{\partial p} + p\,\theta\frac{\partial \mathrm{v}}{\partial\theta}\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}$$

$$(\mathrm{d}Q) = \left(\mu_k - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right)\theta\frac{\partial \mathrm{v}}{\partial\theta} - p\,l_{m_k}\frac{\partial \mathrm{v}}{\partial p}$$

Group 12

θ, $\mathbf{m}_i$, $\mathfrak{n}$ constant.

$$(\partial p) = -\frac{l_{m_k}}{\theta}$$

$$(\partial \mathbf{m}_k) = -\frac{\partial \mathrm{v}}{\partial\theta}$$

$$(\partial \mathrm{v}) = -\frac{l_{m_k}}{\theta}\frac{\partial \mathrm{v}}{\partial p} - \frac{\partial \mathrm{v}}{\partial\theta}\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}$$

$$(\partial\varepsilon) = -\left(\mu_k - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right)\frac{\partial \mathrm{v}}{\partial\theta} + \frac{p}{\theta}l_{m_k}\frac{\partial \mathrm{v}}{\partial p}$$

$$(\partial\zeta) = -\mu_k\frac{\partial \mathrm{v}}{\partial\theta} - \frac{\mathrm{v}}{\theta}l_{m_k}$$

$$(\partial\chi) = -\mu_k\frac{\partial \mathrm{v}}{\partial\theta} - \frac{\mathrm{v}}{\theta}l_{m_k}$$

$$(\partial\psi) = -\left(\mu_k - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right)\frac{\partial \mathrm{v}}{\partial\theta} + \frac{p}{\theta}l_{m_k}\frac{\partial \mathrm{v}}{\partial p}$$

$$(\mathrm{d}W) = p\frac{\partial \mathrm{v}}{\partial\theta}\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} + \frac{p}{\theta}l_{m_k}\frac{\partial \mathrm{v}}{\partial p}.$$

$$(\mathrm{d}Q) = 0$$

Group 13

θ, $\mathbf{m}_i$, ζ constant.

$$(\partial p) = -\mu_k$$

$$(\partial \mathbf{m}_k) = \mathrm{v}$$

$$(\partial \mathrm{v}) = -\mu_k \frac{\partial \mathrm{v}}{\partial p} + \mathrm{v} \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}$$

$$(\partial \varepsilon) = \mu_k \left(\theta \frac{\partial \mathrm{v}}{\partial \theta} + p \frac{\partial \mathrm{v}}{\partial p}\right) + \mathrm{v}\left(\mu_k + l_{m_k} - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right)$$

$$(\partial \mathfrak{n}) = \mu_k \frac{\partial \mathrm{v}}{\partial \theta} + \mathrm{v} \frac{l_{m_k}}{\theta}$$

$$(\partial \chi) = \mu_k \theta \frac{\partial \mathrm{v}}{\partial \theta} + \mathrm{v}\, l_{m_k}$$

$$(\partial \psi) = \mu_k p \frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\left(\mu_k - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right)$$

$$(d\mathbf{W}) = \mu_k p \frac{\partial \mathrm{v}}{\partial p} - p\, \mathrm{v} \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}$$

$$(d\mathbf{Q}) = \mu_k \theta \frac{\partial \mathrm{v}}{\partial \theta} + \mathrm{v}\, l_{m_k}$$

Group 14

θ, $\mathbf{m}_i$, χ constant.

$$(\partial p) = -\mu_k - l_{m_k}$$

$$(\partial \mathbf{m}_k) = -\theta \frac{\partial \mathrm{v}}{\partial \theta} + \mathrm{v}$$

$$(\partial \mathrm{v}) = -(\mu_k + l_{m_k}) \frac{\partial \mathrm{v}}{\partial p} - \left(\theta \frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}\right) \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}$$

$$(\partial \varepsilon) = (\mu_k + l_{m_k})\left(p \frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right) + \left(\theta \frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}\right) p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}$$

$$(\partial \mathfrak{n}) = \mu_k \frac{\partial \mathrm{v}}{\partial \theta} + \frac{\mathrm{v}}{\theta} l_{m_k}$$

$$(\partial \zeta) = -\theta \mu_k \frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}\, l_{m_k}$$

Group 14 (Con.)

$$(\partial\psi) = \mu_k\left(p\frac{\partial v}{\partial p} + v - \theta\frac{\partial v}{\partial\theta}\right) + p\, l_{m_k}\frac{\partial v}{\partial p} + p\frac{\partial v}{\partial m_k}\left(\theta\frac{\partial v}{\partial\theta} - v\right)$$

$$(dW) = (\mu_k + l_{m_k})\, p\frac{\partial v}{\partial p} + p\frac{\partial v}{\partial m_k}\left(\theta\frac{\partial v}{\partial\theta} - v\right)$$

$$(dQ) = \theta\,\mu_k\frac{\partial v}{\partial\theta} + v\, l_{m_k}$$

Group 15

θ, m_i, ψ constant.

$$(\partial p) = -\mu_k + p\frac{\partial v}{\partial m_k}$$

$$(\partial m_k) = -p\frac{\partial v}{\partial p}$$

$$(\partial v) = -\mu_k\frac{\partial v}{\partial p}$$

$$(\partial\varepsilon) = \theta\frac{\partial v}{\partial\theta}\left(\mu_k - p\frac{\partial v}{\partial m_k}\right) - p\frac{\partial v}{\partial p}\, l_{m_k}$$

$$(\partial\eta) = \left(\mu_k - p\frac{\partial v}{\partial m_k}\right)\frac{\partial v}{\partial\theta} - \frac{p}{\theta}\, l_{m_k}\frac{\partial v}{\partial p}.$$

$$(\partial\zeta) = -\mu_k\, p\frac{\partial v}{\partial p} - v\left(\mu_k - p\frac{\partial v}{\partial m_k}\right)$$

$$(\partial\chi) = -\mu_k\left(p\frac{\partial v}{\partial p} + v - \theta\frac{\partial v}{\partial\theta}\right) - p\, l_{m_k}\frac{\partial v}{\partial p} - p\frac{\partial v}{\partial m_k}\left(\theta\frac{\partial v}{\partial\theta} - v\right).$$

$$(dW) = \mu_k\, p\frac{\partial v}{\partial p}$$

$$(dQ) = \left(\mu_k - p\frac{\partial v}{\partial m_k}\right)\theta\frac{\partial v}{\partial\theta} - p\, l_{m_k}\frac{\partial v}{\partial p}.$$

Group 16

p, m_i, v constant.

$$(\partial\theta) = -\frac{\partial v}{\partial m_k}$$

Group 16 (Con.)

$$(\partial \mathbf{m}_k) = \frac{\partial \mathbf{v}}{\partial \theta}$$

$$(\partial \boldsymbol{\varepsilon}) = (\mu_k + l_{m_k}) \frac{\partial \mathbf{v}}{\partial \theta} - c_p (\mathbf{m}_1 + \cdots + \mathbf{m}_n) \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial \mathbf{n}) = \frac{l_{m_k}}{\theta} \frac{\partial \mathbf{v}}{\partial \theta} - (\mathbf{m}_1 + \cdots + \mathbf{m}_n) \frac{c_p}{\theta} \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial \zeta) = \mu_k \frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n} \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial \chi) = (\mu_k + l_{m_k}) \frac{\partial \mathbf{v}}{\partial \theta} - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial \psi) = \mu_k \frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n} \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(d\mathbf{W}) = 0$$

$$(d\mathbf{Q}) = l_{m_k} \frac{\partial \mathbf{v}}{\partial \theta} - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

Group 17

$p, \mathbf{m}_i, \boldsymbol{\varepsilon}$ constant.

$$(\partial \theta) = - l_{m_k} + p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k$$

$$(\partial \mathbf{m}_k) = (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p - p \frac{\partial \mathbf{v}}{\partial \theta}$$

$$(\partial \mathbf{v}) = - (\mu_k + l_{m_k}) \frac{\partial \mathbf{v}}{\partial \theta} + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial \mathbf{n}) = - \frac{p}{\theta} l_{m_k} \frac{\partial \mathbf{v}}{\partial \theta} - (\mathbf{m}_1 + \cdots + \mathbf{m}_n) \frac{c_p}{\theta} \left[\mu_k - p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} \right]$$

$$(\partial \zeta) = \mathbf{n}\, l_{m_k} + \mu_k \left[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p - p \frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n} \right] - \mathbf{n}\, p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial \chi) = - (\mu_k + l_{m_k})\, p \frac{\partial \mathbf{v}}{\partial \theta} + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\, p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

Group 17 (Con.)

$$(\partial\psi) = l_{m_k}\left(\mathfrak{n} + p\frac{\partial \mathrm{v}}{\partial\theta}\right) + \mu_k\left[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p + \mathfrak{n}\right] - \left[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p + \mathfrak{n}\right] p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}$$

$$(d\mathrm{W}) = (\mu_k + l_{m_k})\, p\frac{\partial \mathrm{v}}{\partial\theta} - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\, p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}$$

$$(d\mathbf{Q}) = -\, p\, l_{m_k}\frac{\partial \mathrm{v}}{\partial\theta} - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left[\mu_k - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right]$$

Group 18

$p, \mathbf{m}_i, \mathfrak{n}$ constant.

$$(\partial\theta) = -\frac{l_{m_k}}{\theta}$$

$$(\partial \mathbf{m}_k) = (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}$$

$$(\partial \mathrm{v}) = -\frac{l_{m_k}}{\theta}\frac{\partial \mathrm{v}}{\partial\theta} + \frac{c_p(\mathbf{m}_1 + \cdots + \mathbf{m}_n)}{\theta}\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}$$

$$(\partial\varepsilon) = \frac{p}{\theta}\, l_{m_k}\frac{\partial \mathrm{v}}{\partial\theta} + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_k - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right]$$

$$(\partial\zeta) = \frac{1}{\theta}\left[\mathfrak{n}\, l_{m_k} + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\, \mu_k\right]$$

$$(\partial\chi) = \frac{1}{\theta}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\, \mu_k$$

$$(\partial\psi) = \frac{1}{\theta}\left[l_{m_k}\left(\mathfrak{n} + p\frac{\partial \mathrm{v}}{\partial\theta}\right) + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left(\mu_k - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right)\right]$$

$$(d\mathrm{W}) = p\frac{l_{m_k}}{\theta}\frac{\partial \mathrm{v}}{\partial\theta} - \frac{(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p}{\theta}\, p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}$$

$$(d\mathbf{Q}) = 0$$

Group 19

p, m_i, ζ constant.

$$(\partial\theta) = -\mu_k$$

$$(\partial \mathrm{m}_k) = -\mathfrak{n}$$

$$(\partial \mathrm{v}) = -\mu_k \frac{\partial \mathrm{v}}{\partial \theta} - \mathfrak{n} \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}$$

$$(\partial \varepsilon) = -\mu_k \left[(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p - p \frac{\partial \mathrm{v}}{\partial \theta} + \mathfrak{n} \right] - \mathfrak{n}\, l_{m_k} + \mathfrak{n}\, p \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}$$

$$(\partial \mathfrak{n}) = -\frac{1}{\theta} \left[(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p\, \mu_k + \mathfrak{n}\, l_{m_k} \right]$$

$$(\partial \chi) = -\mu_k (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p - \mathfrak{n}\, (l_{m_k} + \mu_k)$$

$$(\partial \psi) = \mu_k\, p \frac{\partial \mathrm{v}}{\partial \theta} + \mathfrak{n}\, p \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}$$

$$(d\mathrm{W}) = \mu_k\, p \frac{\partial \mathrm{v}}{\partial \theta} + \mathfrak{n}\, p \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}$$

$$(d\mathbf{Q}) = -\left[(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p\, \mu_k + \mathfrak{n}\, l_{m_k} \right]$$

Group 20

p, m_i, χ constant.

$$(\partial\theta) = -\mu_k - l_{m_k}$$

$$(\partial \mathrm{m}_k) = (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p$$

$$(\partial \mathrm{v}) = -(\mu_k + l_{m_k}) \frac{\partial \mathrm{v}}{\partial \theta} + (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}$$

$$(\partial \varepsilon) = (\mu_k + l_{m_k})\, p \frac{\partial \mathrm{v}}{\partial \theta} - (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p\, p \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}$$

$$(\partial \mathfrak{n}) = -\frac{1}{\theta} (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p\, \mu_k$$

$$(\partial \zeta) = \mu_k \left[(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p + \mathfrak{n} \right] + \mathfrak{n}\, l_{m_k}$$

$$(\partial \psi) = (\mu_k + l_{m_k}) \left(\mathfrak{n} + p \frac{\partial \mathrm{v}}{\partial \theta} \right) + (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p \left(\mu_k - p \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k} \right)$$

$$(d\mathrm{W}) = (\mu_k + l_{m_k})\, p \frac{\partial \mathrm{v}}{\partial \theta} - (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p\, p \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}$$

$$(d\mathbf{Q}) = -(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\, c_p\, \mu_k$$

Group 21

$p, \mathbf{m}_i, \psi$ constant.

$$(\partial\theta) = -\mu_k + p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial \mathbf{m}_k) = -p\frac{\partial \mathbf{v}}{\partial\theta} - \mathfrak{n}$$

$$(\partial \mathbf{v}) = -\mu_k\frac{\partial \mathbf{v}}{\partial\theta} - \mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial\varepsilon) = -\mu_k\left[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p + \mathfrak{n}\right] - l_{m_k}\left(\mathfrak{n} + p\frac{\partial \mathbf{v}}{\partial\theta}\right) + \left[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p + \mathfrak{n}\right]p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial\mathfrak{n}) = -\frac{c_p}{\theta}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\left[\mu_k - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] - \frac{l_{m_k}}{\theta}\left(\mathfrak{n} + p\frac{\partial \mathbf{v}}{\partial\theta}\right)$$

$$(\partial\zeta) = -p\left[\mu_k\frac{\partial \mathbf{v}}{\partial\theta} + \mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right]$$

$$(\partial\chi) = -\mu_k\left[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p + \mathfrak{n} + p\frac{\partial \mathbf{v}}{\partial\theta}\right] - l_{m_k}\left[\mathfrak{n} + p\frac{\partial \mathbf{v}}{\partial\theta}\right] + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}p$$

$$(d\mathbf{W}) = p\left[\mu_k\frac{\partial \mathbf{v}}{\partial\theta} + \mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right]$$

$$(d\mathbf{Q}) = -(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left[\mu_k - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] - l_{m_k}\left(\mathfrak{n} + p\frac{\partial \mathbf{v}}{\partial\theta}\right)$$

Group 22

$\mathbf{m}_i, \mathbf{v}, \varepsilon$ constant.

$$(\partial\theta) = -(\mu_k + l_{m_k})\frac{\partial \mathbf{v}}{\partial p} - \theta\frac{\partial \mathbf{v}}{\partial\theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial p) = (\mu_k + l_{m_k})\frac{\partial \mathbf{v}}{\partial\theta} - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}.$$

Group 22 (Con.)

$$(\partial \mathbf{m}_k) = (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \frac{\partial \mathbf{v}}{\partial p} + \theta \left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2$$

$$(\partial \mathfrak{n}) = -\, \mu_k \left[\frac{1}{\theta}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \frac{\partial \mathbf{v}}{\partial p} + \left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2\right]$$

$$(\partial \zeta) = (\mu_k + l_{m_k})\left(\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} + \mathfrak{n}\frac{\partial \mathbf{v}}{\partial p}\right) + \mu_k\Big[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \frac{\partial \mathbf{v}}{\partial p} +$$
$$\theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2\Big] + \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[\mathfrak{n}\,\theta \frac{\partial \mathbf{v}}{\partial \theta} - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\, \mathbf{v}\right].$$

$$(\partial \chi) = \mathbf{v}\left[(\mu_k + l_{m_k})\frac{\partial \mathbf{v}}{\partial \theta} - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right]$$

$$(\partial \psi) = (\mu_k + l_{m_k})\, \mathfrak{n}\frac{\partial \mathbf{v}}{\partial p} + \mu_k\Big[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \frac{\partial \mathbf{v}}{\partial p}$$
$$+ \theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2\Big] + \mathfrak{n}\,\theta\frac{\partial \mathbf{v}}{\partial \theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(d\mathbf{W}) = 0$$

$$(d\mathbf{Q}) = -\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\, \mu_k \frac{\partial \mathbf{v}}{\partial p} - \mu_k\, \theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2$$

Group 23

$\mathbf{m}_i$, $\mathbf{v}$, $\mathfrak{n}$ constant.

$$(\partial \theta) = -\frac{1}{\theta} l_{m_k}\frac{\partial \mathbf{v}}{\partial p} - \frac{\partial \mathbf{v}}{\partial \theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial p) = \frac{1}{\theta} l_{m_k}\frac{\partial \mathbf{v}}{\partial \theta} - \frac{1}{\theta} c_p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)$$

$$(\partial \mathbf{m}_k) = \frac{1}{\theta}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \frac{\partial \mathbf{v}}{\partial p} + \left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2$$

$$(\partial \varepsilon) = \mu_k\left[\frac{1}{\theta}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \frac{\partial \mathbf{v}}{\partial p} + \left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2\right]$$

$$(\partial \zeta) = \mu_k\left[\frac{1}{\theta}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \frac{\partial \mathbf{v}}{\partial p} + \left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2\right] + \mathbf{v}\Big[\frac{l_{m_k}}{\theta}\frac{\partial \mathbf{v}}{\partial \theta}$$
$$- \frac{(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p}{\theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\Big] + \mathfrak{n}\left[\frac{l_{m_k}}{\theta}\frac{\partial \mathbf{v}}{\partial p} + \frac{\partial \mathbf{v}}{\partial \theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right]$$

Group 23 (Con.)

$$(\partial\chi) = \mu_{\rm k}\left[\frac{(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\,c_p}{\theta}\frac{\partial \mathrm{v}}{\partial p} + \left(\frac{\partial \mathrm{v}}{\partial\theta}\right)^2\right] + \mathrm{v}\left[\frac{l_{m_{\rm k}}}{\theta}\frac{\partial \mathrm{v}}{\partial\theta} - \frac{(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\,c_p}{\theta}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\rm k}}\right]$$

$$(\partial\psi) = \mu_{\rm k}\left[\frac{(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\,c_p}{\theta}\frac{\partial \mathrm{v}}{\partial p} + \left(\frac{\partial \mathrm{v}}{\partial\theta}\right)^2\right] + \mathfrak{n}\left[\frac{l_{m_{\rm k}}}{\theta}\frac{\partial \mathrm{v}}{\partial p} + \frac{\partial \mathrm{v}}{\partial\theta}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\rm k}}\right]$$

$$(d\mathbf{W}) = 0$$

$$(d\mathbf{Q}) = 0$$

Group 24

m_i, v, ζ constant.

$$(\partial\theta) = -\mu_{\rm k}\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\rm k}}$$

$$(\partial p) = \mu_{\rm k}\frac{\partial \mathrm{v}}{\partial\theta} + \mathfrak{n}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\rm k}}$$

$$(\partial \mathrm{m}_{\rm k}) = -\mathrm{v}\frac{\partial \mathrm{v}}{\partial\theta} - \mathfrak{n}\frac{\partial \mathrm{v}}{\partial p}$$

$$(\partial\varepsilon) = -(\mu_{\rm k} + l_{m_{\rm k}})\left(\mathrm{v}\frac{\partial \mathrm{v}}{\partial\theta} + \mathfrak{n}\frac{\partial \mathrm{v}}{\partial p}\right) - \mu_{\rm k}\left[(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\,c_p\frac{\partial \mathrm{v}}{\partial p} + \theta\left(\frac{\partial \mathrm{v}}{\partial\theta}\right)^2\right] - \frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\rm k}}\left[\mathfrak{n}\,\theta\frac{\partial \mathrm{v}}{\partial\theta} - (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\,c_p\,\mathrm{v}\right].$$

$$(\partial\mathfrak{n}) = -\mu_{\rm k}\left[(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\frac{c_p}{\theta}\frac{\partial \mathrm{v}}{\partial p} + \left(\frac{\partial \mathrm{v}}{\partial\theta}\right)^2\right] - \frac{l_{m_{\rm k}}}{\theta}\left(\mathrm{v}\frac{\partial \mathrm{v}}{\partial\theta} + \mathfrak{n}\frac{\partial \mathrm{v}}{\partial p}\right) - \frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\rm k}}\left[\mathfrak{n}\frac{\partial \mathrm{v}}{\partial\theta} - \mathrm{v}\,(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\frac{c_p}{\theta}\right].$$

$$(\partial\chi) = -(\mu_{\rm k} + l_{m_{\rm k}})\,\mathfrak{n}\frac{\partial \mathrm{v}}{\partial p} - \mu_{\rm k}\left[c_p\,(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\frac{\partial \mathrm{v}}{\partial p} + \theta\left(\frac{\partial \mathrm{v}}{\partial\theta}\right)^2\right] - l_{m_{\rm k}}\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial\theta} - \frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\rm k}}\left[\mathfrak{n}\,\theta\frac{\partial \mathrm{v}}{\partial\theta} - (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\,c_p\,\mathrm{v} - \mathrm{v}\,\mathfrak{n}\right].$$

Group 24 (Con.)

$$(\partial\psi) = -\mu_k \mathrm{v}\frac{\partial \mathrm{v}}{\partial\theta} - \mathfrak{n}\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}$$

$$(d\mathrm{W}) = 0$$

$$(d\mathrm{Q}) = -\mu_k\left[(\mathrm{m_1}+\cdots+\mathrm{m_n})\,c_p\frac{\partial \mathrm{v}}{\partial p} + \theta\left(\frac{\partial \mathrm{v}}{\partial\theta}\right)^2\right] - l_{m_k}\left(\mathrm{v}\frac{\partial \mathrm{v}}{\partial\theta} + \mathfrak{n}\frac{\partial \mathrm{v}}{\partial p}\right) - \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[\mathfrak{n}\theta\frac{\partial \mathrm{v}}{\partial\theta} - \mathrm{v}(\mathrm{m_1}+\cdots+\mathrm{m_n})\,c_p\right].$$

Group 25

$\mathrm{m_i}$, v, χ constant.

$$(\partial\theta) = -(\mu_k + l_{m_k})\frac{\partial \mathrm{v}}{\partial p} - \left(\theta\frac{\partial \mathrm{v}}{\partial\theta} - \mathrm{v}\right)\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}.$$

$$(\partial p) = (\mu_k + l_{m_k})\frac{\partial \mathrm{v}}{\partial\theta} - (\mathrm{m_1}+\cdots+\mathrm{m_n})\,c_p\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}.$$

$$(\partial \mathrm{m_k}) = -\mathrm{v}\frac{\partial \mathrm{v}}{\partial\theta} + (\mathrm{m_1}+\cdots+\mathrm{m_n})\,c_p\frac{\partial \mathrm{v}}{\partial p} + \theta\left(\frac{\partial \mathrm{v}}{\partial\theta}\right)^2$$

$$(\partial\varepsilon) = -(\mu_k + l_{m_k})\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial\theta} + \mathrm{v}(\mathrm{m_1}+\cdots+\mathrm{m_n})\,c_p\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}.$$

$$(\partial\mathfrak{n}) = -\mu_k\left[(\mathrm{m_1}+\cdots+\mathrm{m_n})\frac{c_p}{\theta}\frac{\partial \mathrm{v}}{\partial p} + \left(\frac{\partial \mathrm{v}}{\partial\theta}\right)^2\right] - \frac{l_{m_k}}{\theta}\mathrm{v}\frac{\partial \mathrm{v}}{\partial\theta} + (\mathrm{m_1}+\cdots+\mathrm{m_n})\frac{c_p}{\theta}\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}.$$

$$(\partial\zeta) = (\mu_k + l_{m_k})\,\mathfrak{n}\frac{\partial \mathrm{v}}{\partial p} + \mu_k\left[c_p(\mathrm{m_1}+\cdots+\mathrm{m_n})\frac{\partial \mathrm{v}}{\partial p} + \theta\left(\frac{\partial \mathrm{v}}{\partial\theta}\right)^2\right] + l_{m_k}\mathrm{v}\frac{\partial \mathrm{v}}{\partial\theta} + \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[\mathfrak{n}\theta\frac{\partial \mathrm{v}}{\partial\theta} - (\mathrm{m_1}+\cdots+\mathrm{m_n})\,c_p\mathrm{v} - \mathrm{v}\,\mathfrak{n}\right].$$

$$(\partial\psi) = \mu_k\left[(\mathrm{m_1}+\cdots+\mathrm{m_n})\,c_p\frac{\partial \mathrm{v}}{\partial p} + \theta\left(\frac{\partial \mathrm{v}}{\partial\theta}\right)^2 + \mathfrak{n}\frac{\partial \mathrm{v}}{\partial p} - \mathrm{v}\frac{\partial \mathrm{v}}{\partial\theta}\right] + l_{m_k}\mathfrak{n}\frac{\partial \mathrm{v}}{\partial p} + \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\mathfrak{n}\left(\theta\frac{\partial \mathrm{v}}{\partial\theta} - \mathrm{v}\right).$$

$$(d\mathrm{W}) = 0$$

$$(d\mathrm{Q}) = -\mu_k\left[(\mathrm{m_1}+\cdots+\mathrm{m_n})\,c_p\frac{\partial \mathrm{v}}{\partial p} + \theta\left(\frac{\partial \mathrm{v}}{\partial\theta}\right)^2\right] - l_{m_k}\mathrm{v}\frac{\partial \mathrm{v}}{\partial\theta} + (\mathrm{m_1}+\cdots+\mathrm{m_n})\,c_p\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}$$

Group 26

$\mathbf{m}_i$, v, ψ constant.

$$(\partial\theta) = -\mu_k \frac{\partial v}{\partial p}.$$

$$(\partial p) = \mu_k \frac{\partial v}{\partial \theta} + \mathfrak{n} \frac{\partial v}{\partial \mathbf{m}_k}.$$

$$(\partial \mathbf{m}_k) = -\mathfrak{n} \frac{\partial v}{\partial p}.$$

$$(\partial \varepsilon) = -(\mu_k + l_{m_k})\,\mathfrak{n}\frac{\partial v}{\partial p} - \mu_k\left[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\frac{\partial v}{\partial p} + \theta\left(\frac{\partial v}{\partial \theta}\right)^2\right] - \mathfrak{n}\,\theta\frac{\partial v}{\partial \theta}\frac{\partial v}{\partial \mathbf{m}_k}.$$

$$(\partial \mathfrak{n}) = -\mu_k\left[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\frac{\partial v}{\partial p} + \left(\frac{\partial v}{\partial \theta}\right)^2\right] - \mathfrak{n}\frac{l_{m_k}}{\theta}\frac{\partial v}{\partial p} - \mathfrak{n}\frac{\partial v}{\partial \theta}\frac{\partial v}{\partial \mathbf{m}_k}.$$

$$(\partial \zeta) = \mu_k\, v \frac{\partial v}{\partial \theta} + \mathfrak{n}\, v \frac{\partial v}{\partial \mathbf{m}_k}$$

$$(\partial \chi) = -(\mu_k + l_{m_k})\,\mathfrak{n}\frac{\partial v}{\partial p} - \mu_k\left[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\frac{\partial v}{\partial p} + \theta\left(\frac{\partial v}{\partial \theta}\right)^2 - v\frac{\partial v}{\partial \theta}\right] - \frac{\partial v}{\partial \mathbf{m}_k}\left[\mathfrak{n}\theta\frac{\partial v}{\partial \theta} - \mathfrak{n}\, v\right].$$

$$(dW) = 0$$

$$(dQ) = -\mu_k\left[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\frac{\partial v}{\partial p} + \theta\left(\frac{\partial v}{\partial \theta}\right)^2\right] - \mathfrak{n}\, l_{m_k}\frac{\partial v}{\partial p} - \mathfrak{n}\,\theta\frac{\partial v}{\partial \theta}\frac{\partial v}{\partial \mathbf{m}_k}$$

Group 27

$\mathbf{m}_i$, ε, $\mathfrak{n}$ constant.

$$(\partial\theta) = -\left(\mu_k - p\frac{\partial v}{\partial \mathbf{m}_k}\right)\frac{\partial v}{\partial \theta} + p\frac{l_{m_k}}{\theta}\frac{\partial v}{\partial p}.$$

$$(\partial p) = -\left(\mu_k - p\frac{\partial v}{\partial \mathbf{m}_k}\right)(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta} - p\frac{l_{m_k}}{\theta}\frac{\partial v}{\partial \theta}$$

Group 27 (Con.)

$$(\partial \mathrm{m_k}) = -p\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2 - p\,(\mathrm{m_1} + \cdots + \mathrm{m_n})\,\frac{c_p}{\theta}\frac{\partial \mathrm{v}}{\partial p}.$$

$$(\partial \mathrm{v}) = -\mu_\mathrm{k}\left[\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2 + (\mathrm{m_1} + \cdots + \mathrm{m_n})\,\frac{c_p}{\theta}\frac{\partial \mathrm{v}}{\partial p}\right]$$

$$(\partial \zeta) = -\mu_\mathrm{k}\left[p\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2 + p\,(\mathrm{m_1} + \cdots + \mathrm{m_n})\,\frac{c_p}{\theta}\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\,(\mathrm{m_1} + \cdots + \mathrm{m_n})\,\frac{c_p}{\theta} - \mathbf{n}\frac{\partial \mathrm{v}}{\partial \theta}\right] - \frac{l_{m_\mathrm{k}}}{\theta}\left[p\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} + p\,\mathbf{n}\frac{\partial \mathrm{v}}{\partial p}\right] - p\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[\mathbf{n}\frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}\,(\mathrm{m_1} + \cdots + \mathrm{m_n})\,\frac{c_p}{\theta}\right]$$

$$(\partial \chi) = -\mu_\mathrm{k}\left[p\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2 + p\,(\mathrm{m_1} + \cdots + \mathrm{m_n})\,\frac{c_p}{\theta}\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\,(\mathrm{m_1} + \cdots + \mathrm{m_n})\,\frac{c_p}{\theta}\right] - p\,\mathrm{v}\frac{l_{m_\mathrm{k}}}{\theta}\frac{\partial \mathrm{v}}{\partial \theta} + \mathrm{v}\,p\,(\mathrm{m_1} + \cdots + \mathrm{m_n})\,\frac{c_p}{\theta}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}$$

$$(\partial \psi) = \mu_\mathrm{k}\,\mathbf{n}\frac{\partial \mathrm{v}}{\partial \theta} - \mathbf{n}\,p\frac{l_{m_\mathrm{k}}}{\theta}\frac{\partial \mathrm{v}}{\partial p} - \mathbf{n}\,p\frac{\partial \mathrm{v}}{\partial \theta}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}$$

$$(d\mathrm{W}) = p\,\mu_\mathrm{k}\left[\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2 + (\mathrm{m_1} + \cdots + \mathrm{m_n})\,\frac{c_p}{\theta}\frac{\partial \mathrm{v}}{\partial p}\right]$$

$$(d\mathbf{Q}) = 0$$

Group 28

$\mathrm{m_i}$, ε, ζ constant.

$$(\partial \theta) = \mu_\mathrm{k}\left(\theta\frac{\partial \mathrm{v}}{\partial \theta} + p\frac{\partial \mathrm{v}}{\partial p}\right) - p\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} + \mathrm{v}\,(\mu_\mathrm{k} + l_{m_\mathrm{k}})$$

$$(\partial p) = \mu_\mathrm{k}\left[(\mathrm{m_1} + \cdots + \mathrm{m_n})\,c_p - p\frac{\partial \mathrm{v}}{\partial \theta} + \mathbf{n}\right] + \mathbf{n}\,l_{m_\mathrm{k}} - p\,\mathbf{n}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}$$

$$(\partial \mathrm{m_k}) = \mathbf{n}\left[\theta\frac{\partial \mathrm{v}}{\partial \theta} + p\frac{\partial \mathrm{v}}{\partial p}\right] - \mathrm{v}\left[(\mathrm{m_1} + \cdots + \mathrm{m_n})\,c_p - p\frac{\partial \mathrm{v}}{\partial \theta}\right]$$

$$(\partial \mathrm{v}) = \mu_\mathrm{k}\left[(\mathrm{m_1} + \cdots + \mathrm{m_n})\,c_p\frac{\partial \mathrm{v}}{\partial p} + \theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2\right] + (\mu_\mathrm{k} + l_{m_\mathrm{k}})\left(\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} + \mathbf{n}\frac{\partial \mathrm{v}}{\partial p}\right) + \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[\mathbf{n}\,\theta\frac{\partial \mathrm{v}}{\partial \theta} - (\mathrm{m_1} + \cdots + \mathrm{m_n})\,c_p\,\mathrm{v}\right]$$

Group 28 (Con.)

$$(\partial \mathbf{n}) = \mu_k \left[p \left(\frac{\partial \mathrm{v}}{\partial \theta} \right)^2 + (\mathrm{m}_1 + \cdots + \mathrm{m}_n) \frac{c_p}{\theta} \left(p \frac{\partial \mathrm{v}}{\partial p} + \mathrm{v} \right) - \mathbf{n} \frac{\partial \mathrm{v}}{\partial \theta} \right]$$
$$+ p \frac{l_{m_k}}{\theta} \left[\mathrm{v} \frac{\partial \mathrm{v}}{\partial \theta} + \mathbf{n} \frac{\partial \mathrm{v}}{\partial p} \right] - p \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k} \left[\mathrm{v} (\mathrm{m}_1 + \cdots + \mathrm{m}_n) \frac{c_p}{\theta} \right.$$
$$\left. - \mathbf{n} \frac{\partial \mathrm{v}}{\partial \theta} \right]$$

$$(\partial \chi) = \mu_k \left[p\, \theta \left(\frac{\partial \mathrm{v}}{\partial \theta} \right)^2 + p (\mathrm{m}_1 + \cdots + \mathrm{m}_n) c_p \frac{\partial \mathrm{v}}{\partial p} + \mathrm{v} (\mathrm{m}_1 + \cdots \right.$$
$$\left. + \mathrm{m}_n) c_p + p\, \mathbf{n} \frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\, \mathbf{n} \right] + l_{m_k} \left[p\, \mathrm{v} \frac{\partial \mathrm{v}}{\partial \theta} + p\, \mathbf{n} \frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\, \mathbf{n} \right]$$
$$+ p \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k} \left[\theta\, \mathbf{n} \frac{\partial \mathrm{v}}{\partial \theta} - \mathbf{n}\, \mathrm{v} - \mathrm{v} (\mathrm{m}_1 + \cdots + \mathrm{m}_n) c_p \right].$$

$$(\partial \psi) = - \mu_k \left[p\, \theta \left(\frac{\partial \mathrm{v}}{\partial \theta} \right)^2 + (\mathrm{m}_1 + \cdots + \mathrm{m}_n) c_p \left(p \frac{\partial \mathrm{v}}{\partial p} + \mathrm{v} \right) \right.$$
$$\left. + p\, \mathbf{n} \frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\, \mathbf{n} \right] - l_{m_k} \left[p\, \mathrm{v} \frac{\partial \mathrm{v}}{\partial \theta} + \mathrm{v}\, \mathbf{n} + p\, \mathbf{n} \frac{\partial \mathrm{v}}{\partial p} \right]$$
$$+ p\, \mathrm{v} \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k} \left[(\mathrm{m}_1 + \cdots + \mathrm{m}_n) c_p + \mathbf{n} \right] - \theta\, p\, \mathbf{n} \frac{\partial \mathrm{v}}{\partial \theta} \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}$$

$$(\mathrm{d}W) = - p\, \mu_k \left[(\mathrm{m}_1 + \cdots + \mathrm{m}_n) c_p \frac{\partial \mathrm{v}}{\partial p} + \theta \left(\frac{\partial \mathrm{v}}{\partial \theta} \right)^2 \right] - p (\mu_k$$
$$+ l_{m_k}) \left(\mathrm{v} \frac{\partial \mathrm{v}}{\partial \theta} + \mathbf{n} \frac{\partial \mathrm{v}}{\partial p} \right) - p \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k} \left[\mathbf{n}\, \theta \frac{\partial \mathrm{v}}{\partial \theta} \right.$$
$$\left. - (\mathrm{m}_1 + \cdots + \mathrm{m}_n) c_p\, \mathrm{v} \right]$$

$$(\mathrm{d}Q) = \mu_k \left[\theta\, p \left(\frac{\partial \mathrm{v}}{\partial \theta} \right)^2 + (\mathrm{m}_1 + \cdots + \mathrm{m}_n) c_p \left(p \frac{\partial \mathrm{v}}{\partial p} + \mathrm{v} \right) \right]$$
$$+ p\, \mathrm{v}\, l_{m_k} \frac{\partial \mathrm{v}}{\partial \theta} - p\, \mathrm{v} (\mathrm{m}_1 + \cdots + \mathrm{m}_n) c_p \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k} - \theta\, \mathbf{n} \frac{\partial \mathrm{v}}{\partial \theta}$$
$$\left[\mu_k - p \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k} \right] + \mathbf{n}\, p\, l_{m_k} \frac{\partial \mathrm{v}}{\partial p}$$

Group 29

$\mathbf{m}_{\mathrm{i}}$, $\boldsymbol{\varepsilon}$, χ constant.

$$(\partial\theta) = (\mu_{\mathrm{k}} + l_{m_{\mathrm{k}}})\left(p\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right) + p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}\left(\theta\frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}\right).$$

$$(\partial p) = -\,(\mu_{\mathrm{k}} + l_{m_{\mathrm{k}}})\,p\frac{\partial \mathrm{v}}{\partial \theta} + p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}(\mathbf{m}_1 + \cdots + \mathbf{m}_{\mathrm{n}})\,c_p.$$

$$(\partial \mathbf{m}_{\mathrm{k}}) = -\,p\frac{\partial \mathrm{v}}{\partial \theta}\left[\theta\frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}\right] - (\mathbf{m}_1 + \cdots + \mathbf{m}_{\mathrm{n}})\,c_p\left[p\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right].$$

$$(\partial \mathrm{v}) = (\mu_{\mathrm{k}} + l_{m_{\mathrm{k}}})\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}(\mathbf{m}_1 + \cdots + \mathbf{m}_{\mathrm{n}})\,c_p.$$

$$(\partial \mathbf{n}) = \mu_{\mathrm{k}}\left[p\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2 + (\mathbf{m}_1 + \cdots + \mathbf{m}_{\mathrm{n}})\frac{c_p}{\theta}\left(p\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right)\right]$$
$$+\,p\,\mathrm{v}\frac{l_{m_{\mathrm{k}}}}{\theta}\frac{\partial \mathrm{v}}{\partial \theta} - p\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}(\mathbf{m}_1 + \cdots + \mathbf{m}_{\mathrm{n}})\frac{c_p}{\theta}.$$

$$(\partial\zeta) = -\,\mu_{\mathrm{k}}\left[p\,\theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2 + (\mathbf{m}_1 + \cdots + \mathbf{m}_{\mathrm{n}})\,c_p\left(p\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right)\right.$$
$$\left. +\,p\,\mathbf{n}\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\,\mathbf{n}\right] - l_{m_{\mathrm{k}}}\left[p\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} + p\,\mathbf{n}\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\,\mathbf{n}\right]$$
$$-\,p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}\left[\theta\,\mathbf{n}\frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}\,\mathbf{n} - \mathrm{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_{\mathrm{n}})\,c_p\right].$$

$$(\partial\psi) = -\,\mu_{\mathrm{k}}\left[p\,\theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2 + (\mathbf{m}_1 + \cdots + \mathbf{m}_{\mathrm{n}})\,c_p\left(p\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right)\right.$$
$$\left. +\,p\,\mathbf{n}\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\,\mathbf{n}\right] - l_{m_{\mathrm{k}}}\left[p\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} + p\,\mathbf{n}\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\,\mathbf{n}\right]$$
$$-\,p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}\left[\theta\,\mathbf{n}\frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}\,\mathbf{n} - \mathrm{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_{\mathrm{n}})\,c_p\right].$$

$$(d\mathbf{W}) = -\,(\mu_{\mathrm{k}} + l_{m_{\mathrm{k}}})\,p\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} + p\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}(\mathbf{m}_1 + \cdots + \mathbf{m}_{\mathrm{n}})\,c_p.$$

$$(d\mathbf{Q}) = \mu_{\mathrm{k}}\left[\theta\,p\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2 + (\mathbf{m}_1 + \cdots + \mathbf{m}_{\mathrm{n}})\,c_p\left(p\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right)\right]$$
$$+\,p\,\mathrm{v}\,l_{m_{\mathrm{k}}}\frac{\partial \mathrm{v}}{\partial \theta} - p\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}(\mathbf{m}_1 + \cdots + \mathbf{m}_{\mathrm{n}})\,c_p.$$

Group 30

$\mathrm{m_i}$, ε, ψ constant.

$$(\partial\theta) = \left(\mu_k - p\frac{\partial v}{\partial m_k}\right)\theta\frac{\partial v}{\partial\theta} - p\,l_{m_k}\frac{\partial v}{\partial p}.$$

$$(\partial p) = \left(\mu_k - p\frac{\partial v}{\partial m_k}\right)\Big[\mathfrak{n} + (m_1 + \cdots + m_n)\,c_p\Big] + l_{m_k}\left(\mathfrak{n} + p\frac{\partial v}{\partial\theta}\right).$$

$$(\partial m_k) = \mathfrak{n}\left[\theta\frac{\partial v}{\partial\theta} + p\frac{\partial v}{\partial p}\right] + p\left[\theta\left(\frac{\partial v}{\partial\theta}\right)^2 + (m_1 + \cdots + m_n)\,c_p\frac{\partial v}{\partial p}\right].$$

$$(\partial v) = (\mu_k + l_{m_k})\,\mathfrak{n}\frac{\partial v}{\partial p} + \mu_k\left[(m_1 + \cdots + m_n)\,c_p\frac{\partial v}{\partial p} + \theta\left(\frac{\partial v}{\partial\theta}\right)^2\right] + \theta\,\mathfrak{n}\frac{\partial v}{\partial\theta}\frac{\partial v}{\partial m_k}.$$

$$(\partial\mathfrak{n}) = -\left(\mu_k - p\frac{\partial v}{\partial m_k}\right)\mathfrak{n}\frac{\partial v}{\partial\theta} + p\,\mathfrak{n}\frac{l_{m_k}}{\theta}\frac{\partial v}{\partial p}$$

$$(\partial\zeta) = \mu_k\left[p\,\theta\left(\frac{\partial v}{\partial\theta}\right)^2 + (m_1 + \cdots + m_n)\,c_p\left(p\frac{\partial v}{\partial p} + v\right) + p\,\mathfrak{n}\frac{\partial v}{\partial p} + v\,\mathfrak{n}\right] + l_{m_k}\left[p\,v\frac{\partial v}{\partial\theta} + v\,\mathfrak{n} + \mathfrak{n}\,p\frac{\partial v}{\partial p}\right] - p\,v\frac{\partial v}{\partial m_k}\Big[(m_1 + \cdots + m_n)\,c_p + \mathfrak{n}\Big] + \theta\,p\,\mathfrak{n}\frac{\partial v}{\partial\theta}\frac{\partial v}{\partial m_k}$$

$$(\partial\chi) = \mu_k\left[p\,\theta\left(\frac{\partial v}{\partial\theta}\right)^2 + (m_1 + \cdots + m_n)\,c_p\left(p\frac{\partial v}{\partial p} + v\right) + p\,\mathfrak{n}\frac{\partial v}{\partial p} + v\,\mathfrak{n}\right] + l_{m_k}\left[p\,v\frac{\partial v}{\partial\theta} + p\,\mathfrak{n}\frac{\partial v}{\partial p} + v\,\mathfrak{n}\right] + p\frac{\partial v}{\partial m_k}\left[\theta\,\mathfrak{n}\frac{\partial v}{\partial\theta} - v\,\mathfrak{n} - v\,(m_1 + \cdots + m_n)\,c_p\right].$$

$$(dW) = -(\mu_k + l_{m_k})\,p\,\mathfrak{n}\frac{\partial v}{\partial p} - p\,\mu_k\left[(m_1 + \cdots + m_n)\,c_p\frac{\partial v}{\partial p} + \theta\left(\frac{\partial v}{\partial\theta}\right)^2\right] - \theta\,p\,\mathfrak{n}\frac{\partial v}{\partial\theta}\frac{\partial v}{\partial m_k}$$

$$(dQ) = -\left(\mu_k - p\frac{\partial v}{\partial m_k}\right)\theta\,\mathfrak{n}\frac{\partial v}{\partial\theta} + p\,\mathfrak{n}\,l_{m_k}\frac{\partial v}{\partial p}.$$

Group 31

$\mathbf{m}_i$, $\mathfrak{n}$, ζ constant.

$$(\partial\theta) = -\mu_k \frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}\frac{l_{m_k}}{\theta}$$

$$(\partial p) = -\frac{1}{\theta}\Big[\mu_k(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p + l_{m_k}\,\mathfrak{n}\Big]$$

$$(\partial \mathbf{m}_k) = \mathrm{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta} - \mathfrak{n}\frac{\partial \mathrm{v}}{\partial \theta}$$

$$(\partial \mathrm{v}) = -\mu_k\left[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\frac{\partial \mathrm{v}}{\partial p} + \left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2\right] - \mathrm{v}\left[\frac{l_{m_k}}{\theta}\frac{\partial \mathrm{v}}{\partial \theta}\right.$$
$$\left. - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right] - \mathfrak{n}\left[\frac{l_{m_k}}{\theta}\frac{\partial \mathrm{v}}{\partial p} + \frac{\partial \mathrm{v}}{\partial \theta}\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right]$$

$$(\partial \varepsilon) = \mu_k\left[p\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2 + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left(p\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right)\right.$$
$$\left. - \mathfrak{n}\frac{\partial \mathrm{v}}{\partial \theta}\right] + \frac{l_{m_k}}{\theta}\left[p\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} + p\,\mathfrak{n}\frac{\partial \mathrm{v}}{\partial p}\right] + p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\left[\mathfrak{n}\frac{\partial \mathrm{v}}{\partial \theta}\right.$$
$$\left. - \mathrm{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\right].$$

$$(\partial \chi) = -\mathfrak{n}\left[\mu_k\frac{\partial \mathrm{v}}{\partial \theta} + \mathrm{v}\frac{l_{m_k}}{\theta}\right]$$

$$(\partial \psi) = \mu_k\left[p\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2 + p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\frac{\partial \mathrm{v}}{\partial p}\right.$$
$$\left. + \mathrm{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\right] + l_{m_k}\left[\frac{1}{\theta}p\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} + \frac{1}{\theta}p\,\mathfrak{n}\frac{\partial \mathrm{v}}{\partial p}\right.$$
$$\left. + \frac{1}{\theta}\mathfrak{n}\,\mathrm{v}\right] + p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\left[\mathfrak{n}\frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\right].$$

$$(d\mathrm{W}) = p\,\mu_k\left[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\frac{\partial \mathrm{v}}{\partial p} + \left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2\right] + p\,\mathrm{v}\left[\frac{l_{m_k}}{\theta}\frac{\partial \mathrm{v}}{\partial \theta}\right.$$
$$\left. - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right] + p\,\mathfrak{n}\left[\frac{l_{m_k}}{\theta}\frac{\partial \mathrm{v}}{\partial p} + \frac{\partial \mathrm{v}}{\partial \theta}\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right].$$

$$(d\mathbf{Q}) = 0.$$

Group 32

m_i, $\mathfrak{n}$, χ constant.

$$(\partial\theta) = -\mu_k \frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}\frac{l_{m_k}}{\theta}.$$

$$(\partial p) = -\mu_k (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\frac{c_p}{\theta}.$$

$$(\partial \mathrm{m}_k) = \mathrm{v}(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\frac{c_p}{\theta}.$$

$$(\partial \mathrm{v}) = -\mu_k\left[\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2 + (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\frac{c_p}{\theta}\frac{\partial \mathrm{v}}{\partial p}\right] - \mathrm{v}\frac{l_{m_k}}{\theta}\frac{\partial \mathrm{v}}{\partial \theta}$$

$$+ \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\frac{c_p}{\theta}.$$

$$(\partial \varepsilon) = \mu_k\left[p\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2 + (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\frac{c_p}{\theta}\left(p\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right)\right]$$

$$+ p\,\mathrm{v}\frac{l_{m_k}}{\theta}\frac{\partial \mathrm{v}}{\partial \theta} - p\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\frac{c_p}{\theta}.$$

$$(\partial \zeta) = \mathfrak{n}\left[\mu_k\frac{\partial \mathrm{v}}{\partial \theta} + \mathrm{v}\frac{l_{m_k}}{\theta}\right]$$

$$(\partial \psi) = \mu_k\left[p\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2 + (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\frac{c_p}{\theta}\left(p\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right) + \mathfrak{n}\frac{\partial \mathrm{v}}{\partial \theta}\right]$$

$$+ \frac{l_{m_k}}{\theta}\left[p\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} + \mathfrak{n}\,\mathrm{v}\right] - p\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\frac{c_p}{\theta}.$$

$$(d\mathrm{W}) = p\,\mu_k\left[\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2 + (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\frac{c_p}{\theta}\frac{\partial \mathrm{v}}{\partial p}\right] + p\,\mathrm{v}\frac{l_{m_k}}{\theta}\frac{\partial \mathrm{v}}{\partial \theta}$$

$$- p\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\frac{c_p}{\theta}$$

$$(d\mathrm{Q}) = 0.$$

Group 33

$\mathbf{m}_i$, $\mathfrak{n}$, ψ constant.

$$(\partial\theta) = -\left(\mu_k - p\frac{\partial v}{\partial m_k}\right)\frac{\partial v}{\partial\theta} + p\frac{l_{m_k}}{\theta}\frac{\partial v}{\partial p}.$$

$$(\partial p) = -\left(\mu_k - p\frac{\partial v}{\partial m_k}\right)(m_1 + \cdots + m_n)\frac{c_p}{\theta} - \frac{l_{m_k}}{\theta}\left(\mathfrak{n} + p\frac{\partial v}{\partial\theta}\right).$$

$$(\partial m_k) = -p\left(\frac{\partial v}{\partial\theta}\right)^2 - p\frac{\partial v}{\partial p}(m_1 + \cdots + m_n)\frac{c_p}{\theta} - \mathfrak{n}\frac{\partial v}{\partial\theta}.$$

$$(\partial v) = -\mu_k\left[\left(\frac{\partial v}{\partial\theta}\right)^2 + \frac{\partial v}{\partial p}(m_1 + \cdots + m_n)\frac{c_p}{\theta}\right] - \mathfrak{n}\frac{l_{m_k}}{\theta}\frac{\partial v}{\partial p}$$
$$- \mathfrak{n}\frac{\partial v}{\partial\theta}\frac{\partial v}{\partial m_k}.$$

$$(\partial\varepsilon) = -\left(\mu_k - p\frac{\partial v}{\partial m_k}\right)\mathfrak{n}\frac{\partial v}{\partial\theta} + p\,\mathfrak{n}\frac{l_{m_k}}{\theta}\frac{\partial v}{\partial p}.$$

$$(\partial\zeta) = -\mu_k\left[p\left(\frac{\partial v}{\partial\theta}\right)^2 + (m_1 + \cdots + m_n)\frac{c_p}{\theta}\left(p\frac{\partial v}{\partial p} + v\right)\right]$$
$$- \frac{l_{m_k}}{\theta}\left[p\,v\frac{\partial v}{\partial\theta} + p\,\mathfrak{n}\frac{\partial v}{\partial p} + \mathfrak{n}\,v\right] - p\frac{\partial v}{\partial m_k}\left[\mathfrak{n}\frac{\partial v}{\partial\theta}\right.$$
$$\left. - v(m_1 + \cdots + m_n)\frac{c_p}{\theta}\right].$$

$$(\partial\chi) = -\mu_k\left[p\left(\frac{\partial v}{\partial\theta}\right)^2 + (m_1 + \cdots + m_n)\frac{c_p}{\theta}\left(p\frac{\partial v}{\partial p} + v\right)\right.$$
$$\left. + \mathfrak{n}\frac{\partial v}{\partial\theta}\right] - \frac{l_{m_k}}{\theta}\left[p\,v\frac{\partial v}{\partial\theta} + \mathfrak{n}\,v\right] +$$
$$p\,v\frac{\partial v}{\partial m_k}(m_1 + \cdots + m_n)\frac{c_p}{\theta}.$$

$$(dW) = p\,\mu_k\left[\left(\frac{\partial v}{\partial\theta}\right)^2 + \frac{\partial v}{\partial p}(m_1 + \cdots + m_n)\frac{c_p}{\theta}\right] + p\,\mathfrak{n}\frac{l_{m_k}}{\theta}\frac{\partial v}{\partial p}$$
$$+ p\,\mathfrak{n}\frac{\partial v}{\partial\theta}\frac{\partial v}{\partial m_k}.$$

$$(dQ) = 0.$$

Group 34

$\mathbf{m}_i$, ζ, χ constant.

$$(\partial\theta) = -\theta\,\mu_k \frac{\partial \mathbf{v}}{\partial \theta} - \mathbf{v}\,l_{m_k}$$

$$(\partial p) = -\mu_k[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p + \mathbf{n}] - \mathbf{n}\,l_{m_k}$$

$$(\partial \mathbf{m}_k) = \mathbf{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p + \mathbf{n}\left(\mathbf{v} - \theta\frac{\partial \mathbf{v}}{\partial \theta}\right)$$

$$\begin{aligned}(\partial \mathbf{v}) = &-(\mu_k + l_{m_k})\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial p} - \mu_k\Bigg[c_p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{\partial \mathbf{v}}{\partial p} \\ &+ \theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2\Bigg] - \mathbf{v}\,l_{m_k}\frac{\partial \mathbf{v}}{\partial \theta} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\Bigg[\mathbf{n}\theta\frac{\partial \mathbf{v}}{\partial \theta} \\ &- (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\,\mathbf{v} - \mathbf{v}\,\mathbf{n}\Bigg].\end{aligned}$$

$$\begin{aligned}(\partial \varepsilon) = \mu_k\Bigg[&\theta\,p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2 + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right) \\ &+ \mathbf{n}\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right)\Bigg] + l_{m_k}\left[p\,\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} + p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\,\mathbf{n}\right] \\ &+ p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[\theta\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta} - \mathbf{v}\,\mathbf{n} - \mathbf{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\right].\end{aligned}$$

$$(\partial \mathbf{n}) = \mathbf{n}\left[\mu_k\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{v}\frac{l_{m_k}}{\theta}\right].$$

$$\begin{aligned}(\partial \psi) = \mu_k\Bigg[&\theta\,p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2 + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right) \\ &+ \mathbf{n}\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right)\Bigg] + l_{m_k}\left[p\,\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} + p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\,\mathbf{n}\right] \\ &+ p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[\theta\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta} - \mathbf{v}\,\mathbf{n} - \mathbf{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\right].\end{aligned}$$

$$\begin{aligned}(dW) = &(\mu_k + l_{m_k})\,p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial p} + p\,\mu_k\Bigg[c_p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{\partial \mathbf{v}}{\partial p} \\ &+ \theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2\Bigg] + p\,\mathbf{v}\,l_{m_k}\frac{\partial \mathbf{v}}{\partial \theta} + p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\Bigg[\mathbf{n}\theta\frac{\partial \mathbf{v}}{\partial \theta} \\ &- (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\,\mathbf{v} - \mathbf{v}\,\mathbf{n}\Bigg].\end{aligned}$$

$$(dQ) = \mathbf{n}\left[\theta\,\mu_k\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{v}\,l_{m_k}\right].$$

Group 35

$\mathbf{m}_i$, ζ, ψ constant.

$$(\partial\theta) = \mu_k\left[p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right] - p\,\mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial p) = -\mu_k\,p\frac{\partial \mathbf{v}}{\partial \theta} - p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}.$$

$$(\partial \mathbf{m}_k) = p\left[\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}\right] + \mathbf{v}\,\mathbf{n}$$

$$(\partial \mathbf{v}) = \mu_k\,\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{v}\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial \boldsymbol{\varepsilon}) = \mu_k\left[p\,\theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2 + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right)\right.$$
$$\left. + p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\,\mathbf{n}\right] + l_{m_k}\left[p\,\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} + p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\,\mathbf{n}\right]$$
$$- p\,\mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p + \mathbf{n}\right] + \theta\,p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial \mathbf{n}) = \mu_k\left[p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2 + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right)\right]$$
$$+ \frac{l_{m_k}}{\theta}\left[p\,\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} + p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial p} + \mathbf{n}\,\mathbf{v}\right] + p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[\mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta}\right.$$
$$\left. - \mathbf{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\right].$$

$$(\partial \chi) = \mu_k\left[\theta\,p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2 + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right)\right.$$
$$\left. + \mathbf{n}\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right)\right] + l_{m_k}\left[p\,\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} + p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\,\mathbf{n}\right]$$
$$+ p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[\theta\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta} - \mathbf{v}\,\mathbf{n} - \mathbf{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\right]$$

$$(d\mathbf{W}) = -p\,\mathbf{v}\left[\mu_k\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right].$$

$$(d\mathbf{Q}) = \mu_k\left[\theta\,p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2 + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right)\right]$$
$$+ l_{m_k}\left[p\,\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} + p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial p} + \mathbf{n}\,\mathbf{v}\right] + p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[\theta\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta}\right.$$
$$\left. - \mathbf{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\right].$$

Group 36

$\mathrm{m_i}$, χ, ψ constant.

$$(\partial\theta) = \mu_k\left[p\frac{\partial v}{\partial p} + v - \theta\frac{\partial v}{\partial\theta}\right] + p\, l_{m_k}\frac{\partial v}{\partial p} + p\frac{\partial v}{\partial m_k}\left[\theta\frac{\partial v}{\partial\theta} - v\right].$$

$$(\partial p) = -(\mu_k + l_{m_k})\left(\mathfrak{n} + p\frac{\partial v}{\partial\theta}\right) - (m_1 + \cdots + m_n)\, c_p\left[\mu_k - p\frac{\partial v}{\partial m_k}\right]$$

$$(\partial m_k) = \mathfrak{n}\left(v - \theta\frac{\partial v}{\partial\theta}\right) + p\left[v\frac{\partial v}{\partial\theta} - \theta\left(\frac{\partial v}{\partial\theta}\right)^2 - (m_1 + \cdots + m_n)\, c_p\frac{\partial v}{\partial p}\right]$$

$$(\partial v) = -(\mu_k + l_{m_k})\,\mathfrak{n}\frac{\partial v}{\partial p} - \mu_k\left[(m_1 + \cdots + m_n)\, c_p\frac{\partial v}{\partial p} + \theta\left(\frac{\partial v}{\partial\theta}\right)^2 - v\frac{\partial v}{\partial\theta}\right] - \frac{\partial v}{\partial m_k}\left[\theta\,\mathfrak{n}\frac{\partial v}{\partial\theta} - v\,\mathfrak{n}\right]$$

$$(\partial\varepsilon) = \mu_k\left[\theta\, p\left(\frac{\partial v}{\partial\theta}\right)^2 + (m_1 + \cdots + m_n)\, c_p\left(p\frac{\partial v}{\partial p} + v\right) + \mathfrak{n}\left(p\frac{\partial v}{\partial p} + v\right)\right] + l_{m_k}\left[p\, v\frac{\partial v}{\partial\theta} + p\,\mathfrak{n}\frac{\partial v}{\partial p} + v\,\mathfrak{n}\right] + p\frac{\partial v}{\partial m_k}\left[\theta\,\mathfrak{n}\frac{\partial v}{\partial\theta} - v\,\mathfrak{n} - v\,(m_1 + \cdots + m_n)\, c_p\right].$$

$$(\partial\mathfrak{n}) = \mu_k\left[p\left(\frac{\partial v}{\partial\theta}\right)^2 + (m_1 + \cdots + m_n)\frac{c_p}{\theta}\left(p\frac{\partial v}{\partial p} + v\right) + \mathfrak{n}\frac{\partial v}{\partial\theta}\right] + \frac{l_{m_k}}{\theta}\left[p\, v\frac{\partial v}{\partial\theta} + \mathfrak{n}\, v\right] - p\, v\frac{\partial v}{\partial m_k}(m_1 + \cdots + m_n)\frac{c_p}{\theta}.$$

$$(\partial\zeta) = -\mu_k\left[\theta\, p\left(\frac{\partial v}{\partial\theta}\right)^2 + (m_1 + \cdots + m_n)\, c_p\left(p\frac{\partial v}{\partial p} + v\right) + \mathfrak{n}\left(p\frac{\partial v}{\partial p} + v\right)\right] - l_{m_k}\left[p\, v\frac{\partial v}{\partial\theta} + p\,\mathfrak{n}\frac{\partial v}{\partial p} + v\,\mathfrak{n}\right] - p\frac{\partial v}{\partial m_k}\left[\theta\,\mathfrak{n}\frac{\partial v}{\partial\theta} - v\,\mathfrak{n} - v\,(m_1 + \cdots + m_n)\, c_p\right].$$

Group 36 (Con.)

$$(dW) = p\,(\mu_k + l_{m_k})\,\mathfrak{n}\frac{\partial v}{\partial p} + p\,\mu_k\left[(m_1 + \cdots + m_n)\,c_p\frac{\partial v}{\partial p} + \theta\left(\frac{\partial v}{\partial \theta}\right)^2 - v\frac{\partial v}{\partial \theta}\right] + p\frac{\partial v}{\partial m_k}\left[\theta\,\mathfrak{n}\frac{\partial v}{\partial \theta} - v\,\mathfrak{n}\right]$$

$$(dQ) = \mu_k\left[\theta\,p\left(\frac{\partial v}{\partial \theta}\right)^2 + (m_1 + \cdots + m_n)\,c_p\left(p\frac{\partial v}{\partial p} + v\right) + \theta\,\mathfrak{n}\frac{\partial v}{\partial \theta}\right] + l_{m_k}\left[p\,v\frac{\partial v}{\partial \theta} + \mathfrak{n}\,v\right] - p\,v\frac{\partial v}{\partial m_k}(m_1 + \cdots + m_n)\,c_p.$$

*Group 37**

θ, p, m_g, v constant.

$$(\partial m_h) = \frac{\partial v}{\partial m_k}$$

$$(\partial m_k) = -\frac{\partial v}{\partial m_h}$$

$$(\partial \varepsilon) = \frac{\partial v}{\partial m_k}[l_{m_h} + \mu_h] - \frac{\partial v}{\partial m_h}[l_{m_k} + \mu_k]$$

$$(\partial \mathfrak{n}) = \frac{\partial v}{\partial m_k}\frac{l_{m_h}}{\theta} - \frac{\partial v}{\partial m_h}\frac{l_{m_k}}{\theta}$$

$$(\partial \zeta) = \frac{\partial v}{\partial m_k}\mu_h - \frac{\partial v}{\partial m_h}\mu_k$$

$$(\partial \chi) = \frac{\partial v}{\partial m_k}[\mu_h + l_{m_h}] - \frac{\partial v}{\partial m_h}[\mu_k + l_{m_k}]$$

$$(\partial \psi) = \frac{\partial v}{\partial m_k}\mu_h - \frac{\partial v}{\partial m_h}\mu_k$$

$$(dW) = 0$$

$$(dQ) = l_{m_h}\frac{\partial v}{\partial m_k} - l_{m_k}\frac{\partial v}{\partial m_h}$$

*m_g denotes all the component masses except m_k and m_h.

*Group 38**

θ, p, $\mathbf{m}_g$, $\boldsymbol{\varepsilon}$ constant.

$$(\partial \mathbf{m}_h) = l_{m_k} - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} + \mu_k$$

$$(\partial \mathbf{m}_k) = -\left[l_{m_h} - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} + \mu_h\right]$$

$$(\partial \mathbf{v}) = \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}[l_{m_k} + \mu_k] - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}[l_{m_h} + \mu_h]$$

$$(\partial \mathfrak{n}) = \frac{l_{m_h}}{\theta}\left[\mu_k - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] - \frac{l_{m_k}}{\theta}\left[\mu_h - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right]$$

$$(\partial \zeta) = \mu_h\left[l_{m_k} - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] - \mu_k\left[l_{m_h} - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right]$$

$$(\partial \chi) = p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k}) - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h})$$

$$(\partial \psi) = l_{m_k}\left(\mu_h - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right) - l_{m_h}\left(\mu_k - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)$$

$$(d\mathbf{W}) = p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}[\mu_h + l_{m_h}] - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k})$$

$$(d\mathbf{Q}) = l_{m_h}\left[\mu_k - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] - l_{m_k}\left[\mu_h - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right]$$

Group 39

θ, p, $\mathbf{m}_g$, $\mathfrak{n}$ constant.

$$(\partial \mathbf{m}_h) = \frac{l_{m_k}}{\theta}$$

$$(\partial \mathbf{m}_k) = -\frac{l_{m_h}}{\theta}$$

$$(\partial \mathbf{v}) = \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\frac{l_{m_k}}{\theta} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\frac{l_{m_h}}{\theta}$$

$$(\partial \boldsymbol{\varepsilon}) = \frac{l_{m_k}}{\theta}\left[\mu_h - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] - \frac{l_{m_h}}{\theta}\left[\mu_k - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right]$$

* $\mathbf{m}_g$ denotes all the component masses except $\mathbf{m}_k$ and $\mathbf{m}_h$.

Group 39 (Con.)

$$(\partial\zeta) = \mu_{\mathrm{h}}\frac{l_{m_{\mathrm{k}}}}{\theta} - \mu_{\mathrm{k}}\frac{l_{m_{\mathrm{h}}}}{\theta}$$

$$(\partial\chi) = \mu_{\mathrm{h}}\frac{l_{m_{\mathrm{k}}}}{\theta} - \mu_{\mathrm{k}}\frac{l_{m_{\mathrm{h}}}}{\theta}$$

$$(\partial\psi) = \frac{l_{m_{\mathrm{k}}}}{\theta}\left[\mu_{\mathrm{h}} - p\frac{\partial\mathbf{v}}{\partial\mathbf{m}_{\mathrm{h}}}\right] - \frac{l_{m_{\mathrm{h}}}}{\theta}\left[\mu_{\mathrm{k}} - p\frac{\partial\mathbf{v}}{\partial\mathbf{m}_{\mathrm{k}}}\right]$$

$$(d\mathbf{W}) = p\frac{\partial\mathbf{v}}{\partial\mathbf{m}_{\mathrm{k}}}\frac{l_{m_{\mathrm{h}}}}{\theta} - p\frac{\partial\mathbf{v}}{\partial\mathbf{m}_{\mathrm{h}}}\frac{l_{m_{\mathrm{k}}}}{\theta}$$

$$(d\mathbf{Q}) = 0$$

Group 40

θ, p, $\mathbf{m}_{\mathrm{g}}$, ζ constant.

$$(\partial\mathbf{m}_{\mathrm{h}}) = \mu_{\mathrm{k}}$$

$$(\partial\mathbf{m}_{\mathrm{k}}) = -\mu_{\mathrm{h}}$$

$$(\partial\mathbf{v}) = \mu_{\mathrm{k}}\frac{\partial\mathbf{v}}{\partial\mathbf{m}_{\mathrm{h}}} - \mu_{\mathrm{h}}\frac{\partial\mathbf{v}}{\partial\mathbf{m}_{\mathrm{k}}}$$

$$(\partial\boldsymbol{\varepsilon}) = \mu_{\mathrm{k}}\left[l_{m_{\mathrm{h}}} - p\frac{\partial\mathbf{v}}{\partial\mathbf{m}_{\mathrm{h}}}\right] - \mu_{\mathrm{h}}\left[l_{m_{\mathrm{k}}} - p\frac{\partial\mathbf{v}}{\partial\mathbf{m}_{\mathrm{k}}}\right]$$

$$(\partial\mathbf{n}) = \mu_{\mathrm{k}}\frac{l_{m_{\mathrm{h}}}}{\theta} - \mu_{\mathrm{h}}\frac{l_{m_{\mathrm{k}}}}{\theta}$$

$$(\partial\chi) = \mu_{\mathrm{k}}\,l_{m_{\mathrm{h}}} - \mu_{\mathrm{h}}\,l_{m_{\mathrm{k}}}$$

$$(\partial\psi) = p\,\mu_{\mathrm{h}}\frac{\partial\mathbf{v}}{\partial\mathbf{m}_{\mathrm{k}}} - p\,\mu_{\mathrm{k}}\frac{\partial\mathbf{v}}{\partial\mathbf{m}_{\mathrm{h}}}$$

$$(d\mathbf{W}) = p\,\mu_{\mathrm{h}}\frac{\partial\mathbf{v}}{\partial\mathbf{m}_{\mathrm{k}}} - p\,\mu_{\mathrm{k}}\frac{\partial\mathbf{v}}{\partial\mathbf{m}_{\mathrm{h}}}$$

$$(d\mathbf{Q}) = \mu_{\mathrm{k}}\,l_{m_{\mathrm{h}}} - \mu_{\mathrm{h}}\,l_{m_{\mathrm{k}}}$$

Group 41

θ, p, $\mathbf{m}_{\mathrm{g}}$, χ constant.

$$(\partial\mathbf{m}_{\mathrm{h}}) = \mu_{\mathrm{k}} + l_{m_{\mathrm{k}}}$$

$$(\partial\mathbf{m}_{\mathrm{k}}) = -\mu_{\mathrm{h}} - l_{m_{\mathrm{h}}}$$

$$(\partial\mathbf{v}) = \frac{\partial\mathbf{v}}{\partial\mathbf{m}_{\mathrm{h}}}(\mu_{\mathrm{k}} + l_{m_{\mathrm{k}}}) - \frac{\partial\mathbf{v}}{\partial\mathbf{m}_{\mathrm{k}}}(\mu_{\mathrm{h}} + l_{m_{\mathrm{h}}})$$

Group 41 (Con.)

$$(\partial \varepsilon) = p \frac{\partial v}{\partial m_k} (\mu_h + l_{m_h}) - p \frac{\partial v}{\partial m_h} (\mu_k + l_{m_k})$$

$$(\partial \eta) = \mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}$$

$$(\partial \zeta) = \mu_h l_{m_k} - \mu_k l_{m_h}$$

$$(\partial \psi) = \mu_h l_{m_k} - \mu_k l_{m_h} + p \frac{\partial v}{\partial m_k} (\mu_h + l_{m_h}) - p \frac{\partial v}{\partial m_h} (\mu_k + l_{m_k})$$

$$(dW) = p \frac{\partial v}{\partial m_k} (\mu_h + l_{m_h}) - p \frac{\partial v}{\partial m_h} (\mu_k + l_{m_k})$$

$$(dQ) = \mu_k l_{m_h} - \mu_h l_{m_k}$$

Group 42

θ, p, m_g, ψ constant.

$$(\partial m_h) = \mu_k - p \frac{\partial v}{\partial m_k}$$

$$(\partial m_k) = - \mu_h + p \frac{\partial v}{\partial m_h}$$

$$(\partial v) = \mu_k \frac{\partial v}{\partial m_h} - \mu_h \frac{\partial v}{\partial m_k}$$

$$(\partial \varepsilon) = l_{m_h} \left(\mu_k - p \frac{\partial v}{\partial m_k} \right) - l_{m_k} \left(\mu_h - p \frac{\partial v}{\partial m_h} \right)$$

$$(\partial \eta) = \frac{l_{m_h}}{\theta} \left(\mu_k - p \frac{\partial v}{\partial m_k} \right) - \frac{l_{m_k}}{\theta} \left(\mu_h - p \frac{\partial v}{\partial m_h} \right)$$

$$(\partial \zeta) = p \mu_k \frac{\partial v}{\partial m_h} - p \mu_h \frac{\partial v}{\partial m_k}$$

$$(\partial \chi) = \mu_k l_{m_h} - \mu_h l_{m_k} + p \frac{\partial v}{\partial m_h} (\mu_k + l_{m_k}) - p \frac{\partial v}{\partial m_k} (\mu_h + l_{m_h})$$

$$(dW) = p \mu_h \frac{\partial v}{\partial m_k} - p \mu_k \frac{\partial v}{\partial m_h}$$

$$(dQ) = l_{m_h} \left(\mu_k - p \frac{\partial v}{\partial m_k} \right) - l_{m_k} \left(\mu_h - p \frac{\partial v}{\partial m_h} \right)$$

Group 43

θ, $\mathrm{m_g}$, v, $\boldsymbol{\varepsilon}$ constant.

$$(\partial p) = \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}(\mu_\mathrm{h} + l_{m_\mathrm{h}}) - \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}(\mu_\mathrm{k} + l_{m_\mathrm{k}})$$

$$(\partial \mathrm{m_h}) = \theta \frac{\partial \mathrm{v}}{\partial \theta}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} + \frac{\partial \mathrm{v}}{\partial p}(\mu_\mathrm{k} + l_{m_\mathrm{k}})$$

$$(\partial \mathrm{m_k}) = -\theta \frac{\partial \mathrm{v}}{\partial \theta}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} - \frac{\partial \mathrm{v}}{\partial p}(\mu_\mathrm{h} + l_{m_\mathrm{h}})$$

$$(\partial \mathfrak{n}) = \frac{\partial \mathrm{v}}{\partial p}\left[\mu_\mathrm{k}\frac{l_{m_\mathrm{h}}}{\theta} - \mu_\mathrm{h}\frac{l_{m_\mathrm{k}}}{\theta}\right] + \frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_\mathrm{k}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} - \mu_\mathrm{h}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right]$$

$$(\partial \zeta) = \frac{\partial \mathrm{v}}{\partial p}\left[\mu_\mathrm{h} l_{m_\mathrm{k}} - \mu_\mathrm{k} l_{m_\mathrm{h}}\right] + \theta\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_\mathrm{h}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_\mathrm{k}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right] +$$
$$\mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}(\mu_\mathrm{h} + l_{m_\mathrm{h}}) - \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}(\mu_\mathrm{k} + l_{m_\mathrm{k}})\right]$$

$$(\partial \chi) = -\mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}(\mu_\mathrm{k} + l_{m_\mathrm{k}}) - \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}(\mu_\mathrm{h} + l_{m_\mathrm{h}})\right]$$

$$(\partial \psi) = \theta\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_\mathrm{h}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_\mathrm{k}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right] + \frac{\partial \mathrm{v}}{\partial p}\left[\mu_\mathrm{h} l_{m_\mathrm{k}} - \mu_\mathrm{k} l_{m_\mathrm{h}}\right]$$

$$(d\mathrm{W}) = 0$$

$$(d\mathrm{Q}) = \frac{\partial \mathrm{v}}{\partial p}\left[\mu_\mathrm{k} l_{m_\mathrm{h}} - \mu_\mathrm{h} l_{m_\mathrm{k}}\right] + \theta\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_\mathrm{k}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} - \mu_\mathrm{h}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right]$$

Group 44

θ, $\mathrm{m_g}$, v, $\mathfrak{n}$ constant.

$$(\partial p) = \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\frac{l_{m_\mathrm{h}}}{\theta} - \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\frac{l_{m_\mathrm{k}}}{\theta}$$

$$(\partial \mathrm{m_h}) = \frac{1}{\theta}\frac{\partial \mathrm{v}}{\partial p} l_{m_\mathrm{k}} + \frac{\partial \mathrm{v}}{\partial \theta}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}$$

$$(\partial \mathrm{m_k}) = -\frac{1}{\theta}\frac{\partial \mathrm{v}}{\partial p} l_{m_\mathrm{h}} - \frac{\partial \mathrm{v}}{\partial \theta}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}$$

$$(\partial \boldsymbol{\varepsilon}) = \frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_\mathrm{h}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_\mathrm{k}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right] + \frac{\partial \mathrm{v}}{\partial p}\left[\mu_\mathrm{h}\frac{l_{m_\mathrm{k}}}{\theta} - \mu_\mathrm{k}\frac{l_{m_\mathrm{h}}}{\theta}\right]$$

Group 44 (Con.)

$$(\partial\zeta) = \frac{\partial v}{\partial\theta}\left[\mu_h \frac{\partial v}{\partial m_k} - \mu_k \frac{\partial v}{\partial m_h}\right] + \frac{\partial v}{\partial p}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] - v\left[\frac{\partial v}{\partial m_h}\frac{l_{m_k}}{\theta} - \frac{\partial v}{\partial m_k}\frac{l_{m_h}}{\theta}\right]$$

$$(\partial\chi) = \frac{\partial v}{\partial\theta}\left[\mu_h \frac{\partial v}{\partial m_k} - \mu_k \frac{\partial v}{\partial m_h}\right] + \frac{\partial v}{\partial p}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] - v\left[\frac{\partial v}{\partial m_h}\frac{l_{m_k}}{\theta} - \frac{\partial v}{\partial m_k}\frac{l_{m_h}}{\theta}\right]$$

$$(\partial\psi) = \frac{\partial v}{\partial\theta}\left[\mu_h \frac{\partial v}{\partial m_k} - \mu_k \frac{\partial v}{\partial m_h}\right] + \frac{\partial v}{\partial p}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right]$$

$$(dW) = 0$$

$$(dQ) = 0$$

Group 45

θ, m_g, v, ζ constant.

$$(\partial p) = \frac{\partial v}{\partial m_k}\mu_h - \frac{\partial v}{\partial m_h}\mu_k$$

$$(\partial m_h) = \mu_k \frac{\partial v}{\partial p} - v\frac{\partial v}{\partial m_k}$$

$$(\partial m_k) = -\mu_h \frac{\partial v}{\partial p} + v\frac{\partial v}{\partial m_h}$$

$$(\partial\varepsilon) = \theta\frac{\partial v}{\partial\theta}\left(\mu_k \frac{\partial v}{\partial m_h} - \mu_h \frac{\partial v}{\partial m_k}\right) + \frac{\partial v}{\partial p}(\mu_k l_{m_h} - \mu_h l_{m_k}) + v\left[\frac{\partial v}{\partial m_h}(\mu_k + l_{m_k}) - \frac{\partial v}{\partial m_k}(\mu_h + l_{m_h})\right]$$

$$(\partial\eta) = \frac{\partial v}{\partial\theta}\left(\mu_k \frac{\partial v}{\partial m_h} - \mu_h \frac{\partial v}{\partial m_k}\right) + \frac{\partial v}{\partial p}\left(\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right) - v\left[\frac{\partial v}{\partial m_k}\frac{l_{m_h}}{\theta} - \frac{\partial v}{\partial m_h}\frac{l_{m_k}}{\theta}\right]$$

$$(\partial\chi) = \theta\frac{\partial v}{\partial\theta}\left(\mu_k \frac{\partial v}{\partial m_h} - \mu_h \frac{\partial v}{\partial m_k}\right) + \frac{\partial v}{\partial p}\left(\mu_k l_{m_h} - \mu_h l_{m_k}\right) - v\left[\frac{\partial v}{\partial m_k} l_{m_h} - \frac{\partial v}{\partial m_h} l_{m_k}\right]$$

Group 45 (Con.)

$$(\partial\psi) = \mathrm{v}\left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}\right]$$

$$(d\mathrm{W}) = 0$$

$$(d\mathrm{Q}) = \theta \frac{\partial \mathrm{v}}{\partial \theta}\left(\mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}\right) + \frac{\partial \mathrm{v}}{\partial p}(\mu_k l_{m_h} - \mu_h l_{m_k}) - \mathrm{v}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m}_k} l_{m_h} - \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h} l_{m_k}\right)$$

Group 46

θ, m_g, v, χ constant.

$$(\partial p) = \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}(\mu_h + l_{m_h}) - \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}(\mu_k + l_{m_k})$$

$$(\partial \mathrm{m}_h) = \frac{\partial \mathrm{v}}{\partial p}(\mu_k + l_{m_k}) + \left(\theta \frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}\right)\frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}$$

$$(\partial \mathrm{m}_k) = -(\mu_h + l_{m_h})\frac{\partial \mathrm{v}}{\partial p} - \left(\theta \frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}\right)\frac{\partial \mathrm{v}}{\partial \mathrm{m}_h}$$

$$(\partial \varepsilon) = -\mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}(\mu_h + l_{m_h}) - \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h}(\mu_k + l_{m_k})\right]$$

$$(\partial \eta) = \frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}\right] + \frac{\partial \mathrm{v}}{\partial p}\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] - \mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}\frac{l_{m_h}}{\theta} - \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h}\frac{l_{m_k}}{\theta}\right]$$

$$(\partial \zeta) = \theta \frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k} - \mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h}\right] + \frac{\partial \mathrm{v}}{\partial p}\left[\mu_h l_{m_k} - \mu_k l_{m_h}\right] - \mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m}_h} l_{m_k} - \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k} l_{m_h}\right]$$

$$(\partial\psi) = \left[\theta \frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}\right]\left[\mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k} - \mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h}\right] + \frac{\partial \mathrm{v}}{\partial p}\left[\mu_h l_{m_k} - \mu_k l_{m_h}\right]$$

$$(d\mathrm{W}) = 0$$

$$(d\mathrm{Q}) = \theta \frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}\right] + \frac{\partial \mathrm{v}}{\partial p}\left[\mu_k l_{m_h} - \mu_h l_{m_k}\right] - \mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m}_k} l_{m_h} - \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h} l_{m_k}\right]$$

Group 47

θ, $\mathbf{m}_g$, v, ψ constant.

$$(\partial p) = \mu_h \frac{\partial v}{\partial m_k} - \mu_k \frac{\partial v}{\partial m_h}$$

$$(\partial m_h) = \mu_k \frac{\partial v}{\partial p}$$

$$(\partial m_k) = -\mu_h \frac{\partial v}{\partial p}$$

$$(\partial \varepsilon) = \theta \frac{\partial v}{\partial \theta}\left[\mu_k \frac{\partial v}{\partial m_h} - \mu_h \frac{\partial v}{\partial m_k}\right] + \frac{\partial v}{\partial p}\left[\mu_k\, l_{m_h} - \mu_h\, l_{m_k}\right]$$

$$(\partial \eta) = \frac{\partial v}{\partial \theta}\left[\mu_k \frac{\partial v}{\partial m_h} - \mu_h \frac{\partial v}{\partial m_k}\right] + \frac{\partial v}{\partial p}\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right]$$

$$(\partial \zeta) = v\left[\mu_h \frac{\partial v}{\partial m_k} - \mu_k \frac{\partial v}{\partial m_h}\right]$$

$$(\partial \chi) = \left[\theta \frac{\partial v}{\partial \theta} - v\right]\left[\mu_k \frac{\partial v}{\partial m_h} - \mu_h \frac{\partial v}{\partial m_k}\right] + \frac{\partial v}{\partial p}\left[\mu_k\, l_{m_h} - \mu_h\, l_{m_k}\right]$$

$$(dW) = 0$$

$$(dQ) = \theta \frac{\partial v}{\partial \theta}\left[\mu_k \frac{\partial v}{\partial m_h} - \mu_h \frac{\partial v}{\partial m_k}\right] + \frac{\partial v}{\partial p}\left[\mu_k\, l_{m_h} - \mu_h\, l_{m_k}\right]$$

Group 48

θ, $\mathbf{m}_g$, ε, η constant.

$$(\partial p) = \frac{l_{m_h}}{\theta}\left[\mu_k - p \frac{\partial v}{\partial m_k}\right] - \frac{l_{m_k}}{\theta}\left[\mu_h - p \frac{\partial v}{\partial m_h}\right].$$

$$(\partial m_h) = \frac{\partial v}{\partial \theta}\left[\mu_k - p \frac{\partial v}{\partial m_k}\right] - p \frac{l_{m_k}}{\theta}\frac{\partial v}{\partial p}.$$

$$(\partial m_k) = -\frac{\partial v}{\partial \theta}\left[\mu_h - p \frac{\partial v}{\partial m_h}\right] + p \frac{l_{m_h}}{\theta}\frac{\partial v}{\partial p}.$$

$$(\partial v) = \frac{\partial v}{\partial \theta}\left[\mu_k \frac{\partial v}{\partial m_h} - \mu_h \frac{\partial v}{\partial m_k}\right] + \frac{\partial v}{\partial p}\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right]$$

$$(\partial \zeta) = p \frac{\partial v}{\partial \theta}\left(\mu_k \frac{\partial v}{\partial m_h} - \mu_h \frac{\partial v}{\partial m_k}\right) + p \frac{\partial v}{\partial p}\left(\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right) +$$
$$v \frac{l_{m_h}}{\theta}\left(\mu_k - p \frac{\partial v}{\partial m_k}\right) - v \frac{l_{m_k}}{\theta}\left(\mu_h - p \frac{\partial v}{\partial m_h}\right)$$

Group 48 (Con.)

$$(\partial \chi) = p\frac{\partial \mathbf{v}}{\partial \theta}\left(\mu_{\mathrm{k}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}} - \mu_{\mathrm{h}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}}\right) + p\frac{\partial \mathbf{v}}{\partial p}\left(\mu_{\mathrm{k}}\frac{l_{m_{\mathrm{h}}}}{\theta} - \mu_{\mathrm{h}}\frac{l_{m_{\mathrm{k}}}}{\theta}\right) + \mathbf{v}\frac{l_{m_{\mathrm{h}}}}{\theta}\left(\mu_{\mathrm{k}} - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}}\right) - \mathbf{v}\frac{l_{m_{\mathrm{k}}}}{\theta}\left(\mu_{\mathrm{h}} - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right)$$

ψ is constant

$$(\mathrm{d}\mathbf{W}) = -p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_{\mathrm{k}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}} - \mu_{\mathrm{h}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}}\right] - p\frac{\partial \mathbf{v}}{\partial p}\left[\mu_{\mathrm{k}}\frac{l_{m_{\mathrm{h}}}}{\theta} - \mu_{\mathrm{h}}\frac{l_{m_{\mathrm{k}}}}{\theta}\right]$$

$$(\mathrm{d}\mathbf{Q}) = 0$$

Group 49

θ, $\mathbf{m}_{\mathrm{g}}$, $\boldsymbol{\varepsilon}$, ζ constant.

$$(\partial p) = \mu_{\mathrm{h}}\left[l_{m_{\mathrm{k}}} - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}}\right] - \mu_{\mathrm{k}}\left[l_{m_{\mathrm{h}}} - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right]$$

$$(\partial \mathbf{m}_{\mathrm{h}}) = -\mu_{\mathrm{k}}\left(\theta\frac{\partial \mathbf{v}}{\partial \theta} + p\frac{\partial \mathbf{v}}{\partial p}\right) + \mathbf{v}\left[p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}} - \mu_{\mathrm{k}} - l_{m_{\mathrm{k}}}\right]$$

$$(\partial \mathbf{m}_{\mathrm{k}}) = \mu_{\mathrm{h}}\left(\theta\frac{\partial \mathbf{v}}{\partial \theta} + p\frac{\partial \mathbf{v}}{\partial p}\right) - \mathbf{v}\left[p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}} - \mu_{\mathrm{h}} - l_{m_{\mathrm{h}}}\right]$$

$$(\partial \mathbf{v}) = \mathbf{v}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}}(\mu_{\mathrm{h}} + l_{m_{\mathrm{h}}}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}}(\mu_{\mathrm{k}} + l_{m_{\mathrm{k}}})\right] + \theta\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_{\mathrm{h}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}} - \mu_{\mathrm{k}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right] + \frac{\partial \mathbf{v}}{\partial p}\left[\mu_{\mathrm{h}}\,l_{m_{\mathrm{k}}} - \mu_{\mathrm{k}}\,l_{m_{\mathrm{h}}}\right].$$

$$(\partial \mathfrak{n}) = p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_{\mathrm{h}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}} - \mu_{\mathrm{k}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right] + p\frac{\partial \mathbf{v}}{\partial p}\left[\mu_{\mathrm{h}}\frac{l_{m_{\mathrm{k}}}}{\theta} - \mu_{\mathrm{k}}\frac{l_{m_{\mathrm{h}}}}{\theta}\right] - \mathbf{v}\frac{l_{m_{\mathrm{h}}}}{\theta}\left(\mu_{\mathrm{k}} - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}}\right) + \mathbf{v}\frac{l_{m_{\mathrm{k}}}}{\theta}\left(\mu_{\mathrm{h}} - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right)$$

$$(\partial \chi) = \mathbf{v}\left[\mu_{\mathrm{h}}\,l_{m_{\mathrm{k}}} - \mu_{\mathrm{k}}\,l_{m_{\mathrm{h}}}\right] + p\,\mathbf{v}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}}l_{m_{\mathrm{h}}} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}}l_{m_{\mathrm{k}}}\right] + p\,\theta\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_{\mathrm{h}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}} - \mu_{\mathrm{k}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right] + p\frac{\partial \mathbf{v}}{\partial p}\left[\mu_{\mathrm{h}}\,l_{m_{\mathrm{k}}} - \mu_{\mathrm{k}}\,l_{m_{\mathrm{h}}}\right]$$

Group 49 (Con.)

$$(\partial\psi) = \theta\, p\,\frac{\partial \mathrm{v}}{\partial\theta}\left[\mu_{\mathrm{k}}\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{h}}} - \mu_{\mathrm{h}}\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}\right] + p\,\frac{\partial \mathrm{v}}{\partial p}\left[\mu_{\mathrm{k}}\, l_{m_{\mathrm{h}}} - \mu_{\mathrm{h}}\, l_{m_{\mathrm{k}}}\right] + \mathrm{v}\, l_{m_{\mathrm{h}}}\left(\mu_{\mathrm{k}} - p\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}\right) - \mathrm{v}\, l_{m_{\mathrm{k}}}\left(\mu_{\mathrm{h}} - p\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right)$$

$$(d\mathrm{W}) = p\,\mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{h}}}\,(\mu_{\mathrm{k}} + l_{m_{\mathrm{k}}}) - \frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}\,(\mu_{\mathrm{h}} + l_{m_{\mathrm{h}}})\right] + p\,\theta\,\frac{\partial \mathrm{v}}{\partial\theta}\left[\mu_{\mathrm{k}}\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{h}}} - \mu_{\mathrm{h}}\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}\right] + p\,\frac{\partial \mathrm{v}}{\partial p}\left[\mu_{\mathrm{k}}\, l_{m_{\mathrm{h}}} - \mu_{\mathrm{h}}\, l_{m_{\mathrm{k}}}\right]$$

$$(d\mathrm{Q}) = \theta\, p\,\frac{\partial \mathrm{v}}{\partial\theta}\left[\mu_{\mathrm{h}}\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}} - \mu_{\mathrm{k}}\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right] + p\,\frac{\partial \mathrm{v}}{\partial p}\left[\mu_{\mathrm{h}}\, l_{m_{\mathrm{k}}} - \mu_{\mathrm{k}}\, l_{m_{\mathrm{h}}}\right] - \mathrm{v}\, l_{m_{\mathrm{h}}}\left(\mu_{\mathrm{k}} - p\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}\right) + \mathrm{v}\, l_{m_{\mathrm{k}}}\left(\mu_{\mathrm{h}} - p\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right)$$

Group 50

θ, $\mathbf{m}_{\mathrm{g}}$, $\boldsymbol{\varepsilon}$, χ constant.

$$(\partial p) = p\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{h}}}\,(\mu_{\mathrm{k}} + l_{m_{\mathrm{k}}}) - p\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}\,(\mu_{\mathrm{h}} + l_{m_{\mathrm{h}}})$$

$$(\partial \mathbf{m}_{\mathrm{h}}) = -\,\theta\left(\frac{\partial \mathrm{v}}{\partial\theta} - \mathrm{v}\right) p\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}} - \left(p\,\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right)(\mu_{\mathrm{k}} + l_{m_{\mathrm{k}}})$$

$$(\partial \mathbf{m}_{\mathrm{k}}) = \theta\left(\frac{\partial \mathrm{v}}{\partial\theta} - \mathrm{v}\right) p\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{h}}} + \left(p\,\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right)(\mu_{\mathrm{h}} + l_{m_{\mathrm{h}}})$$

$$(\partial \mathrm{v}) = \mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}\,(\mu_{\mathrm{h}} + l_{m_{\mathrm{h}}}) - \frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{h}}}\,(\mu_{\mathrm{k}} + l_{m_{\mathrm{k}}})\right]$$

$$(\partial \mathfrak{n}) = p\,\frac{\partial \mathrm{v}}{\partial\theta}\left(\mu_{\mathrm{h}}\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}} - \mu_{\mathrm{k}}\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right) + p\,\frac{\partial \mathrm{v}}{\partial p}\left(\mu_{\mathrm{h}}\,\frac{l_{m_{\mathrm{k}}}}{\theta} - \mu_{\mathrm{k}}\,\frac{l_{m_{\mathrm{h}}}}{\theta}\right) - \mathrm{v}\,\frac{l_{m_{\mathrm{h}}}}{\theta}\left(\mu_{\mathrm{k}} - p\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}\right) + \mathrm{v}\,\frac{l_{m_{\mathrm{k}}}}{\theta}\left(\mu_{\mathrm{h}} - p\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right)$$

$$(\partial\zeta) = p\,\theta\,\frac{\partial \mathrm{v}}{\partial\theta}\left(\mu_{\mathrm{k}}\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{h}}} - \mu_{\mathrm{h}}\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}\right) + p\,\frac{\partial \mathrm{v}}{\partial p}\,(\mu_{\mathrm{k}}\, l_{m_{\mathrm{h}}} - \mu_{\mathrm{h}}\, l_{m_{\mathrm{k}}}) + \mathrm{v}\, l_{m_{\mathrm{h}}}\left(\mu_{\mathrm{k}} - p\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}\right) - \mathrm{v}\, l_{m_{\mathrm{k}}}\left(\mu_{\mathrm{h}} - p\,\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right)$$

Group 50 (Con.)

$$(\partial\psi) = p\,\theta\frac{\partial v}{\partial\theta}\left(\mu_k\frac{\partial v}{\partial m_h} - \mu_h\frac{\partial v}{\partial m_k}\right) + p\frac{\partial v}{\partial p}(\mu_k\,l_{m_h} - \mu_h\,l_{m_k}) +$$
$$v\,l_{m_h}\left(\mu_k - p\frac{\partial v}{\partial m_k}\right) - v\,l_{m_k}\left(\mu_h - p\frac{\partial v}{\partial m_h}\right)$$

$$(dW) = -\,p\,v\left[\frac{\partial v}{\partial m_k}(\mu_h + l_{m_h}) - \frac{\partial v}{\partial m_h}(\mu_k + l_{m_k})\right]$$

$$(dQ) = p\,\theta\frac{\partial v}{\partial\theta}\left(\mu_h\frac{\partial v}{\partial m_k} - \mu_k\frac{\partial v}{\partial m_h}\right) + p\frac{\partial v}{\partial p}(\mu_h\,l_{m_k} - \mu_k\,l_{m_h}) -$$
$$v\,l_{m_h}\left(\mu_k - p\frac{\partial v}{\partial m_k}\right) + v\,l_{m_k}\left(\mu_h - p\frac{\partial v}{\partial m_h}\right)$$

Group 51

θ, m_g, ε, ψ constant.

$$(\partial p) = l_{m_k}\left(\mu_h - p\frac{\partial v}{\partial m_h}\right) - l_{m_h}\left(\mu_k - p\frac{\partial v}{\partial m_k}\right)$$

$$(\partial m_h) = -\,\theta\frac{\partial v}{\partial\theta}\left(\mu_k - p\frac{\partial v}{\partial m_k}\right) + p\,l_{m_k}\frac{\partial v}{\partial p}$$

$$(\partial m_k) = \theta\frac{\partial v}{\partial\theta}\left(\mu_h - p\frac{\partial v}{\partial m_h}\right) - p\,l_{m_h}\frac{\partial v}{\partial p}.$$

$$(\partial v) = \theta\frac{\partial v}{\partial\theta}\left[\mu_h\frac{\partial v}{\partial m_k} - \mu_k\frac{\partial v}{\partial m_h}\right] + \frac{\partial v}{\partial p}\left[\mu_h\,l_{m_k} - \mu_k\,l_{m_h}\right]$$

n is constant.

$$(\partial\zeta) = \theta\,p\frac{\partial v}{\partial\theta}\left[\mu_h\frac{\partial v}{\partial m_k} - \mu_k\frac{\partial v}{\partial m_h}\right] + p\frac{\partial v}{\partial p}\left[\mu_h\,l_{m_k} - \mu_k\,l_{m_h}\right] +$$
$$v\,l_{m_k}\left[\mu_h - p\frac{\partial v}{\partial m_h}\right] - v\,l_{m_h}\left[\mu_k - p\frac{\partial v}{\partial m_k}\right]$$

$$(\partial\chi) = \theta\,p\frac{\partial v}{\partial\theta}\left[\mu_h\frac{\partial v}{\partial m_k} - \mu_k\frac{\partial v}{\partial m_h}\right] + p\frac{\partial v}{\partial p}\left[\mu_h\,l_{m_k} - \mu_k\,l_{m_h}\right] +$$
$$v\,l_{m_k}\left[\mu_h - p\frac{\partial v}{\partial m_h}\right] - v\,l_{m_h}\left[\mu_k - p\frac{\partial v}{\partial m_k}\right]$$

$$(dW) = -\,\theta\,p\frac{\partial v}{\partial\theta}\left[\mu_h\frac{\partial v}{\partial m_k} - \mu_k\frac{\partial v}{\partial m_h}\right] - p\frac{\partial v}{\partial p}\left[\mu_h\,l_{m_k} - \mu_k\,l_{m_h}\right]$$

$$(dQ) = 0.$$

Group 52

θ, $\mathbf{m}_g$, $\mathfrak{n}$, ζ constant.

$$(\partial p) = \mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}$$

$$(\partial \mathbf{m}_h) = -\mu_k \frac{\partial \mathrm{v}}{\partial \theta} - \frac{\mathrm{v}}{\theta} l_{m_k}$$

$$(\partial \mathbf{m}_k) = \mu_h \frac{\partial \mathrm{v}}{\partial \theta} + \frac{\mathrm{v}}{\theta} l_{m_h}$$

$$(\partial \mathrm{v}) = \frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_h \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\right] + \frac{\partial \mathrm{v}}{\partial p}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] + \mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\frac{l_{m_h}}{\theta} - \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\frac{l_{m_k}}{\theta}\right]$$

$$(\partial \varepsilon) = p\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right] + p\frac{\partial \mathrm{v}}{\partial p}\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] + \mathrm{v}\frac{l_{m_h}}{\theta}\left[\mu_k - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right] - \mathrm{v}\frac{l_{m_k}}{\theta}\left[\mu_h - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\right]$$

χ is constant

$$(\partial \psi) = p\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right] + p\frac{\partial \mathrm{v}}{\partial p}\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] + \mathrm{v}\frac{l_{m_h}}{\theta}\left[\mu_k - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right] - \mathrm{v}\frac{l_{m_k}}{\theta}\left[\mu_h - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\right]$$

$$(d\mathbf{W}) = p\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right] + p\frac{\partial \mathrm{v}}{\partial p}\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] + p\,\mathrm{v}\left[\frac{l_{m_k}}{\theta}\frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} - \frac{l_{m_h}}{\theta}\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right]$$

$$(d\mathbf{Q}) = 0$$

Group 53

θ, $\mathbf{m}_g$, $\mathfrak{n}$, χ constant

ζ is constant. Same as *Group 52*.

Group 54

θ, $\mathbf{m}_g$, $\mathfrak{n}$, ψ constant.

ε is constant. Same as *Group 48*.

Group 55

θ, $\mathbf{m}_g$, ζ, χ constant.

$\mathfrak{n}$ is constant. Same as *Group 52*.

Group 56

θ, $\mathbf{m}_g$, ζ, ψ constant.

$$(\partial p) = \mu_h\, p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} - \mu_k\, p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}$$

$$(\partial \mathbf{m}_h) = \mu_k\, p \frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\left[\mu_k - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right]$$

$$(\partial \mathbf{m}_k) = -\mu_h\, p \frac{\partial \mathrm{v}}{\partial p} - \mathrm{v}\left[\mu_h - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\right]$$

$$(\partial \mathrm{v}) = \mathrm{v}\left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right]$$

$$(\partial \varepsilon) = \theta\, p \frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right] + p \frac{\partial \mathrm{v}}{\partial p}\left[\mu_k\, l_{m_h} - \mu_h\, l_{m_k}\right] +$$
$$\mathrm{v}\, l_{m_h}\left(\mu_k - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right) - \mathrm{v}\, l_{m_k}\left(\mu_h - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\right)$$

$$(\partial \mathfrak{n}) = p \frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right] + p \frac{\partial \mathrm{v}}{\partial p}\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] +$$
$$\mathrm{v} \frac{l_{m_h}}{\theta}\left[\mu_k - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right] - \mathrm{v} \frac{l_{m_k}}{\theta}\left[\mu_h - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\right]$$

$$(\partial \chi) = \theta\, p \frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right] + \left(p \frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right)\left[\mu_k\, l_{m_h} -\right.$$
$$\left.\mu_h\, l_{m_k}\right] + p\, \mathrm{v}\left[l_{m_k} \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} - l_{m_h} \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right]$$

$$(d\mathrm{W}) = p\, \mathrm{v}\left[\mu_h \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\right]$$

$$(d\mathbf{Q}) = \theta\, p \frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right] + p \frac{\partial \mathrm{v}}{\partial p}\left[\mu_k\, l_{m_h} - \mu_h\, l_{m_k}\right] +$$
$$l_{m_h}\, \mathrm{v}\left[\mu_k - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right] - l_{m_k}\, \mathrm{v}\left[\mu_h - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\right]$$

Group 57

θ, $\mathbf{m}_g$, χ, ψ constant.

$$(\partial p) = \mu_h l_{m_k} - \mu_k l_{m_h} - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} (\mu_k + l_{m_k}) + p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} (\mu_h + l_{m_h})$$

$$(\partial \mathbf{m}_h) = \mu_k \left[p \frac{\partial \mathrm{v}}{\partial p} + \mathrm{v} - \theta \frac{\partial \mathrm{v}}{\partial \theta} \right] + p \frac{\partial \mathrm{v}}{\partial p} l_{m_k} + p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} \left[\theta \frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v} \right]$$

$$(\partial \mathbf{m}_k) = -\mu_h \left[p \frac{\partial \mathrm{v}}{\partial p} + \mathrm{v} - \theta \frac{\partial \mathrm{v}}{\partial \theta} \right] - p \frac{\partial \mathrm{v}}{\partial p} l_{m_h} - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} \left[\theta \frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v} \right]$$

$$(\partial \mathrm{v}) = \left[\theta \frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v} \right] \left[\mu_h \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} \right] + \frac{\partial \mathrm{v}}{\partial p} \left[\mu_h l_{m_k} - \mu_k l_{m_h} \right]$$

$$(\partial \varepsilon) = \theta p \frac{\partial \mathrm{v}}{\partial \theta} \left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} \right] + p \frac{\partial \mathrm{v}}{\partial p} \left[\mu_k l_{m_h} - \mu_h l_{m_k} \right] +$$
$$\mathrm{v}\, l_{m_h} \left(\mu_k - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} \right) - \mathrm{v}\, l_{m_k} \left(\mu_h - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} \right)$$

$$(\partial \eta) = p \frac{\partial \mathrm{v}}{\partial \theta} \left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} \right] + p \frac{\partial \mathrm{v}}{\partial p} \left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta} \right] +$$
$$\mathrm{v} \frac{l_{m_h}}{\theta} \left(\mu_k - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} \right) - \mathrm{v} \frac{l_{m_k}}{\theta} \left(\mu_h - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} \right)$$

$$(\partial \zeta) = -\theta p \frac{\partial \mathrm{v}}{\partial \theta} \left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} \right] - p \frac{\partial \mathrm{v}}{\partial p} \left[\mu_k l_{m_h} - \mu_h l_{m_k} \right] -$$
$$\mathrm{v}\, l_{m_h} \left(\mu_k - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} \right) + \mathrm{v}\, l_{m_k} \left(\mu_h - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} \right)$$

$$(d\mathbf{W}) = -p \left[\theta \frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v} \right] \left[\mu_h \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} \right] -$$
$$p \frac{\partial \mathrm{v}}{\partial p} \left[\mu_h l_{m_k} - \mu_k l_{m_h} \right]$$

$$(d\mathbf{Q}) = \theta p \frac{\partial \mathrm{v}}{\partial \theta} \left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} \right] + p \frac{\partial \mathrm{v}}{\partial p} \left[\mu_k l_{m_h} - \mu_h l_{m_k} \right] +$$
$$\mathrm{v}\, l_{m_h} \left(\mu_k - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} \right) - \mathrm{v}\, l_{m_k} \left(\mu_h - p \frac{\partial \mathrm{v}}{\partial \mathbf{m}_h} \right)$$

Group 58

p, $\mathbf{m}_g$, $\mathbf{v}$, $\boldsymbol{\varepsilon}$ constant.

$$(\partial\theta) = \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k})$$

$$(\partial \mathbf{m}_h) = \frac{\partial \mathbf{v}}{\partial \theta}(\mu_k + l_{m_k}) - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial \mathbf{m}_k) = -\frac{\partial \mathbf{v}}{\partial \theta}(\mu_h + l_{m_h}) + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}$$

$$(\partial \mathbf{n}) = \frac{\partial \mathbf{v}}{\partial \theta}\left(\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right) - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right]$$

$$(\partial \zeta) = \frac{\partial \mathbf{v}}{\partial \theta}(\mu_h l_{m_k} - \mu_k l_{m_h}) - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathbf{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h})\right]$$

χ is constant.

$$(\partial \psi) = \frac{\partial \mathbf{v}}{\partial \theta}(\mu_h l_{m_k} - \mu_k l_{m_h}) - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathbf{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h})\right]$$

$$(d\mathbf{W}) = 0$$

$$(d\mathbf{Q}) = \frac{\partial \mathbf{v}}{\partial \theta}(\mu_k l_{m_h} - \mu_h l_{m_k}) - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \left[\mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right]$$

Group 59

p, $\mathbf{m}_g$, $\mathbf{v}$, $\mathbf{n}$ constant.

$$(\partial\theta) = \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\frac{l_{m_h}}{\theta} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\frac{l_{m_k}}{\theta}$$

$$(\partial \mathbf{m}_h) = \frac{l_{m_k}}{\theta}\frac{\partial \mathbf{v}}{\partial \theta} - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

Group 59 (Con.)

$$(\partial m_k) = -\frac{l_{m_h}}{\theta}\frac{\partial v}{\partial \theta} + (m_1 + \cdots + m_n)\frac{c_p}{\theta}\frac{\partial v}{\partial m_h}$$

$$(\partial \varepsilon) = \frac{\partial v}{\partial \theta}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] - (m_1 + \cdots + m_n)\frac{c_p}{\theta}\left[\mu_h \frac{\partial v}{\partial m_k} - \mu_k \frac{\partial v}{\partial m_h}\right]$$

$$(\partial \zeta) = \frac{\partial v}{\partial \theta}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] - (m_1 + \cdots + m_n)\frac{c_p}{\theta}\left[\mu_h \frac{\partial v}{\partial m_k} - \mu_k \frac{\partial v}{\partial m_h}\right] + \mathfrak{n}\left[\frac{\partial v}{\partial m_h}\frac{l_{m_k}}{\theta} - \frac{\partial v}{\partial m_k}\frac{l_{m_h}}{\theta}\right]$$

$$(\partial \chi) = \frac{\partial v}{\partial \theta}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] - (m_1 + \cdots + m_n)\frac{c_p}{\theta}\left[\mu_h \frac{\partial v}{\partial m_k} - \mu_k \frac{\partial v}{\partial m_h}\right]$$

$$(\partial \psi) = \frac{\partial v}{\partial \theta}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] - (m_1 + \cdots + m_n)\frac{c_p}{\theta}\left[\mu_h \frac{\partial v}{\partial m_k} - \mu_k \frac{\partial v}{\partial m_h}\right] + \mathfrak{n}\left[\frac{\partial v}{\partial m_h}\frac{l_{m_k}}{\theta} - \frac{\partial v}{\partial m_k}\frac{l_{m_h}}{\theta}\right]$$

$$(dW) = 0$$

$$(dQ) = 0$$

Group 60

p, m_g, v, ζ constant.

$$(\partial \theta) = \mu_h \frac{\partial v}{\partial m_k} - \mu_k \frac{\partial v}{\partial m_h}$$

$$(\partial m_h) = \mu_k \frac{\partial v}{\partial \theta} + \mathfrak{n}\frac{\partial v}{\partial m_k}$$

$$(\partial m_k) = -\mu_h \frac{\partial v}{\partial \theta} - \mathfrak{n}\frac{\partial v}{\partial m_h}$$

Group 60 (Con.)

$$(\partial \varepsilon) = \frac{\partial v}{\partial \theta}\left[\mu_k l_{m_h} - \mu_h l_{m_k}\right] - (m_1 + \cdots + m_n) c_p \left[\mu_k \frac{\partial v}{\partial m_h} - \mu_h \frac{\partial v}{\partial m_k}\right] - n\left[\frac{\partial v}{\partial m_h}(\mu_k + l_{m_k}) - \frac{\partial v}{\partial m_k}(\mu_h + l_{m_h})\right]$$

$$(\partial n) = \frac{\partial v}{\partial \theta}\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] - (m_1 + \cdots + m_n) \frac{c_p}{\theta} \left[\mu_k \frac{\partial v}{\partial m_h} - \mu_h \frac{\partial v}{\partial m_k}\right] - n\left[\frac{\partial v}{\partial m_h}\frac{l_{m_k}}{\theta} - \frac{\partial v}{\partial m_k}\frac{l_{m_h}}{\theta}\right]$$

$$(\partial \chi) = \frac{\partial v}{\partial \theta}\left[\mu_k l_{m_h} - \mu_h l_{m_k}\right] - (m_1 + \cdots + m_n) c_p \left[\mu_k \frac{\partial v}{\partial m_h} - \mu_h \frac{\partial v}{\partial m_k}\right] - n\left[\frac{\partial v}{\partial m_h}(\mu_k + l_{m_k}) - \frac{\partial v}{\partial m_k}(\mu_h + l_{m_h})\right]$$

ψ is constant.

$$(dW) = 0$$

$$(dQ) = \frac{\partial v}{\partial \theta}\left[\mu_k l_{m_h} - \mu_h l_{m_k}\right] - (m_1 + \cdots + m_n) c_p \left[\mu_k \frac{\partial v}{\partial m_h} - \mu_h \frac{\partial v}{\partial m_k}\right] - n\left[\frac{\partial v}{\partial m_h} l_{m_k} - \frac{\partial v}{\partial m_k} l_{m_h}\right]$$

Group 61

p, m_g, v, χ constant.

ε is constant. Same as *Group 58.*

Group 62

p, m_g, v, ψ constant.

ζ is constant. Same as *Group 60.*

Group 63

p, m_g, ε, n constant.

$$(\partial \theta) = \frac{l_{m_h}}{\theta}\left[\mu_k - p\frac{\partial v}{\partial m_k}\right] - \frac{l_{m_k}}{\theta}\left[\mu_h - p\frac{\partial v}{\partial m_h}\right]$$

$$(\partial m_h) = - p\frac{\partial v}{\partial \theta}\frac{l_{m_k}}{\theta} - (m_1 + \cdots + m_n)\frac{c_p}{\theta}\left[\mu_k - p\frac{\partial v}{\partial m_k}\right]$$

Group 63 (Con.)

$$(\partial \mathrm{m_k}) = p \frac{\partial \mathrm{v}}{\partial \theta} \frac{l_{m_h}}{\theta} + (\mathrm{m_1} + \cdots + \mathrm{m_n}) \frac{c_p}{\theta} \left[\mu_h - p \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} \right]$$

$$(\partial \mathrm{v}) = \frac{\partial \mathrm{v}}{\partial \theta} \left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta} \right] - (\mathrm{m_1} + \cdots + \mathrm{m_n}) \frac{c_p}{\theta} \left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} \right]$$

$$(\partial \zeta) = p \frac{\partial \mathrm{v}}{\partial \theta} \left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta} \right] - p\, (\mathrm{m_1} + \cdots + \mathrm{m_n}) \frac{c_p}{\theta} \left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} \right] + \mathfrak{n} \left[\frac{l_{m_k}}{\theta} \left(\mu_h - p \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} \right) - \frac{l_{m_h}}{\theta} \left(\mu_k - p \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} \right) \right]$$

$$(\partial \chi) = p \frac{\partial \mathrm{v}}{\partial \theta} \left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta} \right] - p\, (\mathrm{m_1} + \cdots + \mathrm{m_n}) \frac{c_p}{\theta} \left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} \right]$$

$$(\partial \psi) = \mathfrak{n} \left[\frac{l_{m_k}}{\theta} \left(\mu_h - p \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} \right) - \frac{l_{m_h}}{\theta} \left(\mu_k - p \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} \right) \right]$$

$$(d\mathbf{W}) = p \frac{\partial \mathrm{v}}{\partial \theta} \left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta} \right] + p\, (\mathrm{m_1} + \cdots + \mathrm{m_n}) \frac{c_p}{\theta} \left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} \right]$$

$$(d\mathbf{Q}) = 0$$

Group 64

p, $\mathrm{m_g}$, ε, ζ constant.

$$(\partial \theta) = \mu_h \left[l_{m_k} - p \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} \right] - \mu_k \left[l_{m_h} - p \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} \right]$$

$$(\partial \mathrm{m_h}) = \mathfrak{n} \left[l_{m_k} - p \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} \right] + \mu_k \left[(\mathrm{m_1} + \cdots + \mathrm{m_n})\, c_p - p \frac{\partial \mathrm{v}}{\partial \theta} + \mathfrak{n} \right]$$

Group 64 (Con.)

$$(\partial \mathbf{m}_k) = -\mathbf{n}\left[l_{m_h} - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] - \mu_h\left[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p - p\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n}\right]$$

$$(\partial \mathbf{v}) = \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h\, l_{m_k} - \mu_k\, l_{m_h}\right] - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathbf{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h})\right]$$

$$(\partial \mathbf{n}) = p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] - p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathbf{n}\left[\frac{l_{m_h}}{\theta}\left(\mu_k - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right) - \frac{l_{m_k}}{\theta}\left(\mu_h - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)\right]$$

$$(\partial \chi) = p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h\, l_{m_k} - \mu_k\, l_{m_h}\right] - p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathbf{n}\, p\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h})\right]$$

$$(\partial \psi) = p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k\, l_{m_h} - \mu_h\, l_{m_k}\right] - p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left[\mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] + p\,\mathbf{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k})\right]$$

$$(d\mathbf{W}) = p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k\, l_{m_h} - \mu_h\, l_{m_k}\right] - p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left[\mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] + p\,\mathbf{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k})\right]$$

$$(d\mathbf{Q}) = p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h\, l_{m_k} - \mu_k\, l_{m_h}\right] - p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathbf{n}\left[l_{m_h}\left(\mu_k - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right) - l_{m_k}\left(\mu_h - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)\right]$$

Group 65

p, $\mathbf{m}_g$, $\boldsymbol{\varepsilon}$, χ constant.

$\mathbf{v}$ is constant. Same as *Group 58.*

Group 66

p, $\mathbf{m}_g$, $\boldsymbol{\varepsilon}$, ψ constant.

$$(\partial\theta) = l_{m_k}\left(\mu_h - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\right) - l_{m_h}\left(\mu_k - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right)$$

$$(\partial \mathbf{m}_h) = l_{m_k}\left(\mathfrak{n} + p\frac{\partial \mathrm{v}}{\partial \theta}\right) + \left(\mu_k - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right)\Big[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p + \mathfrak{n}\Big]$$

$$(\partial \mathbf{m}_k) = -\,l_{m_h}\left(\mathfrak{n} + p\frac{\partial \mathrm{v}}{\partial \theta}\right) - \left(\mu_h - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\right)\Big[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p + \mathfrak{n}\Big]$$

$$(\partial \mathrm{v}) = \frac{\partial \mathrm{v}}{\partial \theta}(\mu_h\, l_{m_k} - \mu_k\, l_{m_h}) - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left[\mu_h\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} - \mu_k\frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\right] + \mathfrak{n}\left[\frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k}) - \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h})\right]$$

$$(\partial \mathfrak{n}) = \mathfrak{n}\left[\frac{l_{m_h}}{\theta}\left(\mu_k - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right) - \frac{l_{m_k}}{\theta}\left(\mu_h - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\right)\right]$$

$$(\partial \zeta) = p\frac{\partial \mathrm{v}}{\partial \theta}(\mu_h\, l_{m_k} - \mu_k\, l_{m_h}) - p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left[\mu_h\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} - \mu_k\frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\right] + \mathfrak{n}\,p\left[\frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k}) - \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h})\right]$$

$$(\partial \chi) = p\frac{\partial \mathrm{v}}{\partial \theta}(\mu_h\, l_{m_k} - \mu_k\, l_{m_h}) - p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left[\mu_h\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} - \mu_k\frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\right] + \mathfrak{n}\,p\left[\frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k}) - \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h})\right]$$

$$(d\mathrm{W}) = -\,p\frac{\partial \mathrm{v}}{\partial \theta}(\mu_h\, l_{m_k} - \mu_k\, l_{m_h}) + p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left[\mu_h\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k} - \mu_k\frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\right] - \mathfrak{n}\,p\left[\frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k}) - \frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h})\right]$$

$$(d\mathrm{Q}) = \mathfrak{n}\left[l_{m_h}\left(\mu_k - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_k}\right) - l_{m_k}\left(\mu_h - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\right)\right]$$

Group 67

p, $\mathbf{m}_g$, $\mathfrak{n}$, ζ constant.

$$(\partial\theta) = \mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}$$

$$(\partial\mathbf{m}_h) = \mathfrak{n}\frac{l_{m_k}}{\theta} + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\mu_k$$

$$(\partial\mathbf{m}_k) = -\mathfrak{n}\frac{l_{m_h}}{\theta} - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\mu_h$$

$$(\partial\mathbf{v}) = \frac{\partial\mathbf{v}}{\partial\theta}\left(\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right) - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_h \frac{\partial\mathbf{v}}{\partial\mathbf{m}_k} - \mu_k \frac{\partial\mathbf{v}}{\partial\mathbf{m}_h}\right] + \mathfrak{n}\left[\frac{\partial\mathbf{v}}{\partial\mathbf{m}_h}\frac{l_{m_k}}{\theta} - \frac{\partial\mathbf{v}}{\partial\mathbf{m}_k}\frac{l_{m_h}}{\theta}\right]$$

$$(\partial\boldsymbol{\varepsilon}) = p\frac{\partial\mathbf{v}}{\partial\theta}\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] - p(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_k \frac{\partial\mathbf{v}}{\partial\mathbf{m}_h} - \mu_h \frac{\partial\mathbf{v}}{\partial\mathbf{m}_k}\right] + \mathfrak{n}\left[\frac{l_{m_k}}{\theta}\left(\mu_h - p\frac{\partial\mathbf{v}}{\partial\mathbf{m}_h}\right) - \frac{l_{m_h}}{\theta}\left(\mu_k - p\frac{\partial\mathbf{v}}{\partial\mathbf{m}_k}\right)\right]$$

$$(\partial\chi) = \mathfrak{n}\left[\frac{l_{m_k}}{\theta}\mu_h - \frac{l_{m_h}}{\theta}\mu_k\right].$$

$$(\partial\psi) = p\frac{\partial\mathbf{v}}{\partial\theta}\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] - p(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_k \frac{\partial\mathbf{v}}{\partial\mathbf{m}_h} - \mu_h \frac{\partial\mathbf{v}}{\partial\mathbf{m}_k}\right] - \mathfrak{n}\,p\left[\frac{l_{m_k}}{\theta}\frac{\partial\mathbf{v}}{\partial\mathbf{m}_h} - \frac{l_{m_h}}{\theta}\frac{\partial\mathbf{v}}{\partial\mathbf{m}_k}\right]$$

$$(d\mathbf{W}) = p\frac{\partial\mathbf{v}}{\partial\theta}\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] - p(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_k \frac{\partial\mathbf{v}}{\partial\mathbf{m}_h} - \mu_h \frac{\partial\mathbf{v}}{\partial\mathbf{m}_k}\right] + p\,\mathfrak{n}\left[\frac{\partial\mathbf{v}}{\partial\mathbf{m}_k}\frac{l_{m_h}}{\theta} - \frac{\partial\mathbf{v}}{\partial\mathbf{m}_h}\frac{l_{m_k}}{\theta}\right]$$

$$(d\mathbf{Q}) = 0$$

Group 68

p, $\mathbf{m}_g$, $\mathfrak{n}$, χ constant.

$$(\partial\theta) = \mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}$$

$$(\partial\mathbf{m}_h) = \mu_k(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}$$

Group 68 (Con.)

$$(\partial \mathrm{m_k}) = -\mu_h (\mathrm{m_1} + \cdots + \mathrm{m_n}) \frac{c_p}{\theta}$$

$$(\partial \mathrm{v}) = \frac{\partial \mathrm{v}}{\partial \theta}\left(\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right) - (\mathrm{m_1} + \cdots + \mathrm{m_n}) \frac{c_p}{\theta}\left[\mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right]$$

$$(\partial \varepsilon) = p \frac{\partial \mathrm{v}}{\partial \theta}\left(\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right) - p (\mathrm{m_1} + \cdots + \mathrm{m_n}) \frac{c_p}{\theta}\left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right]$$

$$(\partial \zeta) = \mathfrak{n}\left(\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right)$$

$$(\partial \psi) = \left(\mathfrak{n} + p \frac{\partial \mathrm{v}}{\partial \theta}\right)\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] + p (\mathrm{m_1} + \cdots + \mathrm{m_n}) \frac{c_p}{\theta}\left[\mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right]$$

$$(d\mathrm{W}) = p \frac{\partial \mathrm{v}}{\partial \theta}\left(\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right) + p (\mathrm{m_1} + \cdots + \mathrm{m_n}) \frac{c_p}{\theta}\left[\mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right]$$

$$(d\mathbf{Q}) = 0$$

Group 69

p, $\mathrm{m_g}$, $\mathfrak{n}$, ψ constant.

$$(\partial \theta) = \left(\mu_h - p \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)\frac{l_{m_k}}{\theta} - \left(\mu_k - p \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)\frac{l_{m_h}}{\theta}$$

$$(\partial \mathrm{m_h}) = \left(\mathfrak{n} + p \frac{\partial \mathrm{v}}{\partial \theta}\right)\frac{l_{m_k}}{\theta} + (\mathrm{m_1} + \cdots + \mathrm{m_n}) \frac{c_p}{\theta}\left(\mu_k - p \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)$$

$$(\partial \mathrm{m_k}) = -\left(\mathfrak{n} + p \frac{\partial \mathrm{v}}{\partial \theta}\right)\frac{l_{m_h}}{\theta} - (\mathrm{m_1} + \cdots + \mathrm{m_n}) \frac{c_p}{\theta}\left(\mu_h - p \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)$$

Group 69 (Con.)

$$(\partial \mathbf{v}) = \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_{\mathrm{h}}\frac{l_{m_{\mathrm{k}}}}{\theta} - \mu_{\mathrm{k}}\frac{l_{m_{\mathrm{h}}}}{\theta}\right] + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_{\mathrm{k}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}} - \mu_{\mathrm{h}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}}\right] + \mathbf{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}}\frac{l_{m_{\mathrm{k}}}}{\theta} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}}\frac{l_{m_{\mathrm{h}}}}{\theta}\right]$$

$$(\partial \boldsymbol{\varepsilon}) = \mathbf{n}\left[\frac{l_{m_{\mathrm{k}}}}{\theta}\left(\mu_{\mathrm{h}} - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right) - \frac{l_{m_{\mathrm{h}}}}{\theta}\left(\mu_{\mathrm{k}} - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}}\right)\right]$$

$$(\partial \zeta) = p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_{\mathrm{h}}\frac{l_{m_{\mathrm{k}}}}{\theta} - \mu_{\mathrm{k}}\frac{l_{m_{\mathrm{h}}}}{\theta}\right] - p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_{\mathrm{h}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}} - \mu_{\mathrm{k}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right] + \mathbf{n}\,p\left[\frac{l_{m_{\mathrm{k}}}}{\theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}} - \frac{l_{m_{\mathrm{h}}}}{\theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}}\right]$$

$$(\partial \chi) = \left(\mathbf{n} + p\frac{\partial \mathbf{v}}{\partial \theta}\right)\left[\mu_{\mathrm{h}}\frac{l_{m_{\mathrm{k}}}}{\theta} - \mu_{\mathrm{k}}\frac{l_{m_{\mathrm{h}}}}{\theta}\right] + p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_{\mathrm{k}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}} - \mu_{\mathrm{h}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}}\right]$$

$$(d\mathbf{W}) = p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_{\mathrm{k}}\frac{l_{m_{\mathrm{h}}}}{\theta} - \mu_{\mathrm{h}}\frac{l_{m_{\mathrm{k}}}}{\theta}\right] + p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_{\mathrm{h}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}} - \mu_{\mathrm{k}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right] + p\,\mathbf{n}\left[\frac{l_{m_{\mathrm{h}}}}{\theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}} - \frac{l_{m_{\mathrm{k}}}}{\theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right]$$

$$(d\mathbf{Q}) = 0$$

Group 70

p, $\mathbf{m}_g$, ζ, χ constant.

$$(\partial \theta) = \mu_{\mathrm{k}}\,l_{m_{\mathrm{h}}} - \mu_{\mathrm{h}}\,l_{m_{\mathrm{k}}}$$

$$(\partial \mathbf{m}_{\mathrm{h}}) = -\,\mathbf{n}\,(\mu_{\mathrm{k}} + l_{m_{\mathrm{k}}}) - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\,\mu_{\mathrm{k}}$$

$$(\partial \mathbf{m}_{\mathrm{k}}) = \mathbf{n}\,(\mu_{\mathrm{h}} + l_{m_{\mathrm{h}}}) + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\,\mu_{\mathrm{h}}$$

$$(\partial \mathbf{v}) = \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_{\mathrm{k}}\,l_{m_{\mathrm{h}}} - \mu_{\mathrm{h}}\,l_{m_{\mathrm{k}}}\right] + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left[\mu_{\mathrm{h}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}} - \mu_{\mathrm{k}}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right] + \mathbf{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}}(\mu_{\mathrm{h}} + l_{m_{\mathrm{h}}}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{h}}}(\mu_{\mathrm{k}} + l_{m_{\mathrm{k}}})\right]$$

Group 70 (Con.)

$$(\partial \varepsilon) = p \frac{\partial v}{\partial \theta}\left[\mu_h l_{m_k} - \mu_k l_{m_h}\right] + p\,(m_1 + \cdots + m_n)\, c_p \left[\mu_k \frac{\partial v}{\partial m_h} - \mu_h \frac{\partial v}{\partial m_k}\right] + p\,\mathbf{n}\left[\frac{\partial v}{\partial m_h}(\mu_k + l_{m_k}) - \frac{\partial v}{\partial m_k}(\mu_h + l_{m_h})\right]$$

$$(\partial \mathbf{n}) = \mathbf{n}\left[\frac{l_{m_k}}{\theta}\mu_h - \frac{l_{m_h}}{\theta}\mu_k\right]$$

$$(\partial \psi) = p \frac{\partial v}{\partial \theta}\left[\mu_h l_{m_k} - \mu_k l_{m_h}\right] + p\,(m_1 + \cdots + m_n)\, c_p \left[\mu_k \frac{\partial v}{\partial m_h} - \mu_h \frac{\partial v}{\partial m_k}\right] + p\,\mathbf{n}\left[\frac{\partial v}{\partial m_h}(\mu_k + l_{m_k}) - \frac{\partial v}{\partial m_k}(\mu_h + l_{m_h})\right]$$

$$(dW) = p \frac{\partial v}{\partial \theta}\left[\mu_h l_{m_k} - \mu_k l_{m_h}\right] + p\,(m_1 + \cdots + m_n)\, c_p \left[\mu_k \frac{\partial v}{\partial m_h} - \mu_h \frac{\partial v}{\partial m_k}\right] + p\,\mathbf{n}\left[\frac{\partial v}{\partial m_h}(\mu_k + l_{m_k}) - \frac{\partial v}{\partial m_k}(\mu_h + l_{m_h})\right]$$

$$(dQ) = \mathbf{n}\left[\mu_h l_{m_k} - \mu_k l_{m_h}\right]$$

Group 71

p, m_g, ζ, ψ constant.

v is constant. Same as *Group 60.*

Group 72

p, m_g, χ, ψ constant.

$$(\partial \theta) = \mu_h l_{m_k} - \mu_k l_{m_h} - p \frac{\partial v}{\partial m_h}(\mu_k + l_{m_k}) + p \frac{\partial v}{\partial m_k}(\mu_h + l_{m_h})$$

$$(\partial m_h) = \left(\mathbf{n} + p \frac{\partial v}{\partial \theta}\right)(\mu_k + l_{m_k}) + (m_1 + \cdots + m_n)\, c_p \left(\mu_k - p \frac{\partial v}{\partial m_k}\right)$$

Group 72 (Con.)

$$(\partial \mathbf{m}_k) = -\left(\mathfrak{n} + p\frac{\partial \mathbf{v}}{\partial \theta}\right)(\mu_h + l_{m_h}) - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left(\mu_h - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)$$

$$(\partial \mathbf{v}) = \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h\, l_{m_k} - \mu_k\, l_{m_h}\right] - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] - \mathfrak{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k})\right]$$

$$(\partial \varepsilon) = p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k\, l_{m_h} - \mu_h\, l_{m_k}\right] + p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + p\,\mathfrak{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k})\right]$$

$$(\partial \mathfrak{n}) = \left(\mathfrak{n} + p\frac{\partial \mathbf{v}}{\partial \theta}\right)\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] + p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left(\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)$$

$$(\partial \zeta) = p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h\, l_{m_k} - \mu_k\, l_{m_h}\right] - p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left(\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right) + p\,\mathfrak{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h})\right]$$

$$(d\mathbf{W}) = p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k\, l_{m_h} - \mu_h\, l_{m_k}\right] + p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + p\,\mathfrak{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k})\right]$$

$$(d\mathbf{Q}) = \left(\mathfrak{n} + p\frac{\partial \mathbf{v}}{\partial \theta}\right)\left[\mu_k\, l_{m_h} - \mu_h\, l_{m_k}\right] + p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left(\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)$$

Group 73

$\mathbf{m}_g$, $\mathbf{v}$, $\boldsymbol{\varepsilon}$, $\mathbf{n}$ constant.

$$(\partial\theta) = \frac{\partial \mathrm{v}}{\partial p}\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] + \frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}\right]$$

$$(\partial p) = \frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] + (\mathrm{m}_1 + \cdots + \mathrm{m}_n)\frac{c_p}{\theta}\left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}\right]$$

$$(\partial \mathrm{m}_h) = -\left[(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\frac{c_p}{\theta}\frac{\partial \mathrm{v}}{\partial p} + \left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2\right]\mu_k$$

$$(\partial \mathrm{m}_k) = \left[(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\frac{c_p}{\theta}\frac{\partial \mathrm{v}}{\partial p} + \left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2\right]\mu_h$$

$$(\partial\zeta) = \left[\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} + \mathrm{n}\frac{\partial \mathrm{v}}{\partial p}\right]\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] - \left[\mathrm{n}\frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\frac{c_p}{\theta}\right]\left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}\right]$$

$$(\partial\chi) = \mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] + \mathrm{v}(\mathrm{m}_1 + \cdots + \mathrm{m}_n)\frac{c_p}{\theta}\left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}\right]$$

$$(\partial\psi) = \mathrm{n}\frac{\partial \mathrm{v}}{\partial p}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] - \mathrm{n}\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h} - \mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}\right]$$

$$(d\mathrm{W}) = 0$$

$$(\partial \mathrm{Q}) = 0$$

Group 74

$\mathbf{m}_g$, $\mathbf{v}$, $\boldsymbol{\varepsilon}$, ζ constant.

$$(\partial\theta) = \theta\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k} - \mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h}\right] + \frac{\partial \mathrm{v}}{\partial p}\left[\mu_h l_{m_k} - \mu_k l_{m_h}\right] + \mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}(\mu_h + l_{m_h}) - \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h}(\mu_k + l_{m_k})\right]$$

$$(\partial p) = \frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_k l_{m_h} - \mu_h l_{m_k}\right] + (\mathrm{m}_1 + \cdots + \mathrm{m}_n)c_p\left[\mu_h \frac{\partial \mathrm{v}}{\partial \mathrm{m}_k} - \mu_k \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h}\right] + \mathrm{n}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}(\mu_h + l_{m_h}) - \frac{\partial \mathrm{v}}{\partial \mathrm{m}_h}(\mu_k + l_{m_k})\right]$$

Group 74 (Con.)

$$(\partial \mathbf{m_h}) = \left(\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}\right)(\mu_k + l_{m_k}) + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \left[\mu_k \frac{\partial \mathbf{v}}{\partial p} - \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m_k}}\right] + \theta \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m_k}}\right]$$

$$(\partial \mathbf{m_k}) = -\left(\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}\right)(\mu_h + l_{m_h}) - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \left[\mu_h \frac{\partial \mathbf{v}}{\partial p} - \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m_h}}\right] - \theta \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m_h}}\right]$$

$$(\partial \mathbf{n}) = \left[\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}\right]\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] + \left[\mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta} - \mathbf{v}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\right]\left[\mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m_h}} - \mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m_k}}\right]$$

$$(\partial \chi) = \mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k l_{m_h} - \mu_h l_{m_k}\right] + \mathbf{v}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m_k}} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m_h}}\right] + \mathbf{v}\,\mathbf{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m_k}}(\mu_h + l_{m_h}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m_h}}(\mu_k + l_{m_k})\right]$$

$$(\partial \psi) = \mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h l_{m_k} - \mu_k l_{m_h}\right] + \mathbf{v}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \left[\mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m_h}} - \mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m_k}}\right] + \mathbf{v}\,\mathbf{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m_h}}(\mu_k + l_{m_k}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m_k}}(\mu_h + l_{m_h})\right]$$

$$(d\mathbf{W}) = 0$$

$$(d\mathbf{Q}) = \left[\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}\right]\left[\mu_k l_{m_h} - \mu_h l_{m_k}\right] + \left[\theta\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta} - \mathbf{v}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\right]\left[\mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m_h}} - \mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m_k}}\right]$$

Group 75

$\mathbf{m_g}$, $\mathbf{v}$, $\boldsymbol{\varepsilon}$, χ constant.

p is constant. Same as *Group 58.*

Group 76

$\mathbf{m}_g$, $\mathbf{v}$, $\boldsymbol{\varepsilon}$, ψ constant.

$$(\partial\theta) = \theta \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \frac{\partial \mathbf{v}}{\partial p}\left[\mu_h l_{m_k} - \mu_k l_{m_h}\right].$$

$$(\partial p) = \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k l_{m_h} - \mu_h l_{m_k}\right] + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathbf{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k})\right].$$

$$(\partial \mathbf{m}_h) = \theta \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \frac{\partial \mathbf{v}}{\partial p}\mu_k + \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}(\mu_k + l_{m_k}).$$

$$(\partial \mathbf{m}_k) = -\,\theta \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p \frac{\partial \mathbf{v}}{\partial p}\mu_h - \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}(\mu_h + l_{m_h}).$$

$$(\partial \mathbf{n}) = \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] + \mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right].$$

$$(\partial\zeta) = \mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k l_{m_h} - \mu_h l_{m_k}\right] + \mathbf{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathbf{v}\,\mathbf{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k})\right]$$

$$(\partial\chi) = \mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k l_{m_h} - \mu_h l_{m_k}\right] + \mathbf{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathbf{v}\,\mathbf{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k})\right]$$

$$(d\mathbf{W}) = 0$$

$$(d\mathbf{Q}) = \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}\left[\mu_k l_{m_h} - \mu_h l_{m_k}\right] + \theta\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right].$$

Group 77

$\mathbf{m}_g$, $\mathbf{v}$, $\mathfrak{n}$, ζ constant.

$$(\partial\theta) = \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \frac{\partial \mathbf{v}}{\partial p}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] - \mathbf{v}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\frac{l_{m_k}}{\theta} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\frac{l_{m_h}}{\theta}\right]$$

$$(\partial p) = \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathfrak{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\frac{l_{m_h}}{\theta} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\frac{l_{m_k}}{\theta}\right]$$

$$(\partial \mathbf{m}_h) = \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{v}\frac{l_{m_k}}{\theta}\right] + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_k \frac{\partial \mathbf{v}}{\partial p} - \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] + \mathfrak{n}\left[\frac{l_{m_k}}{\theta}\frac{\partial \mathbf{v}}{\partial p} + \frac{\partial \mathbf{v}}{\partial \theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right]$$

$$(\partial \mathbf{m}_k) = -\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{v}\frac{l_{m_h}}{\theta}\right] - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial p} - \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] - \mathfrak{n}\left[\frac{l_{m_h}}{\theta}\frac{\partial \mathbf{v}}{\partial p} + \frac{\partial \mathbf{v}}{\partial \theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right]$$

$$(\partial \varepsilon) = \left[\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} + \mathfrak{n}\frac{\partial \mathbf{v}}{\partial p}\right]\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] - \left[\mathfrak{n}\frac{\partial \mathbf{v}}{\partial \theta} - \mathbf{v}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\right]\left[\mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right]$$

$$(\partial \chi) = \mathfrak{n}\frac{\partial \mathbf{v}}{\partial p}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] + \mathfrak{n}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathfrak{n}\,\mathbf{v}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\frac{l_{m_h}}{\theta} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\frac{l_{m_k}}{\theta}\right]$$

$$(\partial \psi) = \mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] + \mathbf{v}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] + \mathfrak{n}\,\mathbf{v}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\frac{l_{m_k}}{\theta} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\frac{l_{m_h}}{\theta}\right]$$

$$(d\mathbf{W}) = 0$$

$$(d\mathbf{Q}) = 0$$

Group 78

$\mathbf{m}_g$, $\mathbf{v}$, $\mathbf{n}$, χ constant.

$$(\partial\theta) = \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \frac{\partial \mathbf{v}}{\partial p}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] - \mathbf{v}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\frac{l_{m_k}}{\theta} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\frac{l_{m_h}}{\theta}\right]$$

$$(\partial p) = \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right]$$

$$(\partial \mathbf{m}_h) = \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{v}\frac{l_{m_k}}{\theta}\right] + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_k \frac{\partial \mathbf{v}}{\partial p} - \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right]$$

$$(\partial \mathbf{m}_k) = -\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{v}\frac{l_{m_h}}{\theta}\right] - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial p} - \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right]$$

$$(\partial \varepsilon) = \mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] + \mathbf{v}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right]$$

$$(\partial \zeta) = \mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] + \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] + \mathbf{n}\,\mathbf{v}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\frac{l_{m_k}}{\theta} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\frac{l_{m_h}}{\theta}\right]$$

$$(\partial \psi) = \left[\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} - \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}\right]\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right] - \left[\mathbf{v}(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta}\right]\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathbf{n}\,\mathbf{v}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\frac{l_{m_k}}{\theta} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\frac{l_{m_h}}{\theta}\right]$$

$$(d\mathbf{W}) = 0$$

$$(d\mathbf{Q}) = 0$$

Group 79

$\mathbf{m}_g$, $\mathbf{v}$, $\mathbf{n}$, ψ constant.

$$(\partial\theta) = \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \frac{\partial \mathbf{v}}{\partial p}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right]$$

$$(\partial p) = \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathbf{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\frac{l_{m_h}}{\theta} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\frac{l_{m_k}}{\theta}\right]$$

$$(\partial \mathbf{m}_h) = \frac{\partial \mathbf{v}}{\partial \theta}\left[\frac{\partial \mathbf{v}}{\partial \theta}\mu_k + \mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\frac{\partial \mathbf{v}}{\partial p}\mu_k + \mathbf{n}\frac{l_{m_k}}{\theta}\frac{\partial \mathbf{v}}{\partial p}$$

$$(\partial \mathbf{m}_k) = -\frac{\partial \mathbf{v}}{\partial \theta}\left[\frac{\partial \mathbf{v}}{\partial \theta}\mu_h + \mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\frac{\partial \mathbf{v}}{\partial p}\mu_h - \mathbf{n}\frac{l_{m_h}}{\theta}\frac{\partial \mathbf{v}}{\partial p}$$

$$(\partial \varepsilon) = \mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}\left[\mu_h \frac{l_{m_k}}{\theta} - \mu_k \frac{l_{m_h}}{\theta}\right]$$

$$(\partial \zeta) = \mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] + \mathbf{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathbf{n}\,\mathbf{v}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\frac{l_{m_h}}{\theta} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\frac{l_{m_k}}{\theta}\right]$$

$$(\partial \chi) = \left[\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta} - \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}\right]\left[\mu_k \frac{l_{m_h}}{\theta} - \mu_h \frac{l_{m_k}}{\theta}\right] + \left[\mathbf{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta}\right]\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathbf{n}\,\mathbf{v}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\frac{l_{m_h}}{\theta} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\frac{l_{m_k}}{\theta}\right]$$

$$(d\mathbf{W}) = 0$$

$$(d\mathbf{Q}) = 0$$

Group 80

$\mathrm{m_g}$, v, ζ, χ constant.

$$(\partial\theta) = \theta \frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_\mathrm{k} \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} - \mu_\mathrm{h} \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right] + \frac{\partial \mathrm{v}}{\partial p}\left[\mu_\mathrm{k}\, l_{m_\mathrm{h}} - \mu_\mathrm{h}\, l_{m_\mathrm{k}}\right] - \mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} l_{m_\mathrm{h}} - \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} l_{m_\mathrm{k}}\right]$$

$$(\partial p) = \frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_\mathrm{h}\, l_{m_\mathrm{k}} - \mu_\mathrm{k}\, l_{m_\mathrm{h}}\right] + (\mathrm{m_1} + \cdots + \mathrm{m_n})\, c_p\left[\mu_\mathrm{k} \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} - \mu_\mathrm{h} \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right] + \mathbf{n}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}(\mu_\mathrm{k} + l_{m_\mathrm{k}}) - \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}(\mu_\mathrm{h} + l_{m_\mathrm{h}})\right]$$

$$(\partial \mathrm{m_h}) = -\theta \frac{\partial \mathrm{v}}{\partial \theta}\left[\frac{\partial \mathrm{v}}{\partial \theta}\mu_\mathrm{k} + \mathrm{v}\frac{l_{m_\mathrm{k}}}{\theta} + \mathbf{n}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right] + (\mathrm{m_1} + \cdots + \mathrm{m_n})\, c_p\left[\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_\mathrm{k}\frac{\partial \mathrm{v}}{\partial p}\right] + \mathbf{n}\left[\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \frac{\partial \mathrm{v}}{\partial p}(\mu_\mathrm{k} + l_{m_\mathrm{k}})\right]$$

$$(\partial \mathrm{m_k}) = \theta \frac{\partial \mathrm{v}}{\partial \theta}\left[\frac{\partial \mathrm{v}}{\partial \theta}\mu_\mathrm{h} + \mathrm{v}\frac{l_{m_\mathrm{h}}}{\theta} + \mathbf{n}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right] - (\mathrm{m_1} + \cdots + \mathrm{m_n})\, c_p\left[\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} - \mu_\mathrm{h}\frac{\partial \mathrm{v}}{\partial p}\right] - \mathbf{n}\left[\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} - \frac{\partial \mathrm{v}}{\partial p}(\mu_\mathrm{h} + l_{m_\mathrm{h}})\right]$$

$$(\partial \varepsilon) = \mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_\mathrm{k}\, l_{m_\mathrm{h}} - \mu_\mathrm{h}\, l_{m_\mathrm{k}}\right] + \mathrm{v}\,(\mathrm{m_1} + \cdots + \mathrm{m_n})\, c_p\left[\mu_\mathrm{h}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_\mathrm{k}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right] + \mathrm{v}\,\mathbf{n}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}(\mu_\mathrm{h} + l_{m_\mathrm{h}}) - \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}(\mu_\mathrm{k} + l_{m_\mathrm{k}})\right]$$

$$(\partial \mathbf{n}) = \mathbf{n}\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_\mathrm{h}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_\mathrm{k}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right] + \mathbf{n}\frac{\partial \mathrm{v}}{\partial p}\left[\mu_\mathrm{h}\frac{l_{m_\mathrm{k}}}{\theta} - \mu_\mathrm{k}\frac{l_{m_\mathrm{h}}}{\theta}\right] + \mathbf{n}\,\mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\frac{l_{m_\mathrm{h}}}{\theta} - \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\frac{l_{m_\mathrm{k}}}{\theta}\right]$$

$$(\partial \psi) = \mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_\mathrm{k}\, l_{m_\mathrm{h}} - \mu_\mathrm{h}\, l_{m_\mathrm{k}}\right] + \mathrm{v}\,(\mathrm{m_1} + \cdots + \mathrm{m_n})\, c_p\left[\mu_\mathrm{h}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_\mathrm{k}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right] + \mathrm{v}\,\mathbf{n}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}(\mu_\mathrm{h} + l_{m_\mathrm{h}}) - \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}(\mu_\mathrm{k} + l_{m_\mathrm{k}})\right]$$

$$(d\mathrm{W}) = 0$$

Group 80 (Con.)

$$(\partial \mathbf{Q}) = \theta\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}\left[\mu_h\, l_{m_k} - \mu_k\, l_{m_h}\right] +$$

$$\mathbf{n}\,\mathbf{v}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} l_{m_h} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} l_{m_k}\right]$$

Group 81

$\mathbf{m}_g$, $\mathbf{v}$, ζ, ψ constant.

p is constant. Same as *Group 60.*

Group 82

$\mathbf{m}_g$, $\mathbf{v}$, χ, ψ constant.

$$(\partial \theta) = \left(\theta\frac{\partial \mathbf{v}}{\partial \theta} - \mathbf{v}\right)\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \frac{\partial \mathbf{v}}{\partial p}\left[\mu_h\, l_{m_k} - \mu_k\, l_{m_h}\right]$$

$$(\partial p) = \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k\, l_{m_h} - \mu_h\, l_{m_k}\right] + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \right.$$

$$\left.\mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathbf{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k})\right]$$

$$(\partial \mathbf{m}_h) = \left(\theta\frac{\partial \mathbf{v}}{\partial \theta} - \mathbf{v}\right)\left[\mu_k \frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] + \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}\left[\mu_k + l_{m_k}\right] +$$

$$(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\, \mu_k \frac{\partial \mathbf{v}}{\partial p}$$

$$(\partial \mathbf{m}_k) = -\left(\theta\frac{\partial \mathbf{v}}{\partial \theta} - \mathbf{v}\right)\left[\mu_h \frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] - \mathbf{n}\frac{\partial \mathbf{v}}{\partial p}\left[\mu_h + l_{m_h}\right] -$$

$$(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\, \mu_h \frac{\partial \mathbf{v}}{\partial p}$$

$$(\partial \varepsilon) = \mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h\, l_{m_k} - \mu_k\, l_{m_h}\right] + \mathbf{v}\left[(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p + \right.$$

$$\left.\mathbf{n}\right]\left[\mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] + \mathbf{v}\,\mathbf{n}\left[l_{m_k}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - l_{m_h}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right]$$

Group 82 (Con.)

$$(\partial \mathfrak{n}) = \left[\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} - \mathfrak{n}\frac{\partial \mathrm{v}}{\partial p}\right]\left[\mu_{\mathrm{h}}\frac{l_{m_{\mathrm{k}}}}{\theta} - \mu_{\mathrm{k}}\frac{l_{m_{\mathrm{h}}}}{\theta}\right] - \left[\mathrm{v}\,(\mathrm{m}_1 + \cdots + \mathrm{m}_{\mathrm{n}})\frac{c_p}{\theta} + \mathfrak{n}\frac{\partial \mathrm{v}}{\partial \theta}\right]\left[\mu_{\mathrm{h}}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{k}}} - \mu_{\mathrm{k}}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{h}}}\right] + \mathfrak{n}\,\mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{h}}}\frac{l_{m_{\mathrm{k}}}}{\theta} - \frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{k}}}\frac{l_{m_{\mathrm{h}}}}{\theta}\right]$$

$$(\partial \zeta) = \mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_{\mathrm{k}}\,l_{m_{\mathrm{h}}} - \mu_{\mathrm{h}}\,l_{m_{\mathrm{k}}}\right] + \mathrm{v}\,(\mathrm{m}_1 + \cdots + \mathrm{m}_{\mathrm{n}})\,c_p\left[\mu_{\mathrm{h}}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{k}}} - \mu_{\mathrm{k}}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{h}}}\right] + \mathfrak{n}\,\mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{k}}}(\mu_{\mathrm{h}} + l_{m_{\mathrm{h}}}) - \frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{h}}}(\mu_{\mathrm{k}} + l_{m_{\mathrm{k}}})\right]$$

$$(\mathrm{d}W) = 0$$

$$(\mathrm{d}\mathbf{Q}) = \left[\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} - \mathfrak{n}\frac{\partial \mathrm{v}}{\partial p}\right]\left[\mu_{\mathrm{h}}\,l_{m_{\mathrm{k}}} - \mu_{\mathrm{k}}\,l_{m_{\mathrm{h}}}\right] - \left[\mathrm{v}\,(\mathrm{m}_1 + \cdots + \mathrm{m}_{\mathrm{n}})\,c_p + \theta\,\mathfrak{n}\frac{\partial \mathrm{v}}{\partial \theta}\right]\left[\mu_{\mathrm{h}}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{k}}} - \mu_{\mathrm{k}}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{h}}}\right] + \mathfrak{n}\,\mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{h}}}\,l_{m_{\mathrm{k}}} - \frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{k}}}\,l_{m_{\mathrm{h}}}\right]$$

Group 83

m_g, ε, $\mathfrak{n}$, ζ constant.

$$(\partial \theta) = p\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_{\mathrm{k}}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{h}}} - \mu_{\mathrm{h}}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{k}}}\right] + p\frac{\partial \mathrm{v}}{\partial p}\left[\mu_{\mathrm{k}}\frac{l_{m_{\mathrm{h}}}}{\theta} - \mu_{\mathrm{h}}\frac{l_{m_{\mathrm{k}}}}{\theta}\right] + \mathrm{v}\frac{l_{m_{\mathrm{h}}}}{\theta}\left(\mu_{\mathrm{k}} - p\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{k}}}\right) - \mathrm{v}\frac{l_{m_{\mathrm{k}}}}{\theta}\left(\mu_{\mathrm{h}} - p\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{h}}}\right)$$

$$(\partial p) = p\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_{\mathrm{h}}\frac{l_{m_{\mathrm{k}}}}{\theta} - \mu_{\mathrm{k}}\frac{l_{m_{\mathrm{h}}}}{\theta}\right] + p\,(\mathrm{m}_1 + \cdots + \mathrm{m}_{\mathrm{n}})\frac{c_p}{\theta}\left[\mu_{\mathrm{k}}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{h}}} - \mu_{\mathrm{h}}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{k}}}\right] + \mathfrak{n}\left[\frac{l_{m_{\mathrm{h}}}}{\theta}\left(\mu_{\mathrm{k}} - p\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{k}}}\right) - \frac{l_{m_{\mathrm{k}}}}{\theta}\left(\mu_{\mathrm{h}} - p\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{h}}}\right)\right]$$

$$(\partial \mathrm{m}_{\mathrm{h}}) = \mu_{\mathrm{k}}\frac{\partial \mathrm{v}}{\partial \theta}\left(\mathfrak{n} - p\frac{\partial \mathrm{v}}{\partial \theta}\right) - (\mathrm{m}_1 + \cdots + \mathrm{m}_{\mathrm{n}})\frac{c_p}{\theta}\left[\mu_{\mathrm{k}}\left(p\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right) - p\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{k}}}\right] - p\frac{l_{m_{\mathrm{k}}}}{\theta}\left[\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} + \mathfrak{n}\frac{\partial \mathrm{v}}{\partial p}\right] - p\,\mathfrak{n}\frac{\partial \mathrm{v}}{\partial \theta}\frac{\partial \mathrm{v}}{\partial \mathrm{m}_{\mathrm{k}}}$$

Group 83 (Con.)

$$(\partial \mathrm{m_k}) = \mu_h \frac{\partial \mathrm{v}}{\partial \theta}\left(p\frac{\partial \mathrm{v}}{\partial \theta} - \mathfrak{n}\right) + (\mathrm{m_1} + \cdots + \mathrm{m_n})\frac{c_p}{\theta}\left[\mu_h\left(p\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right) - p\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right] + p\frac{l_{m_h}}{\theta}\left[\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} + \mathfrak{n}\frac{\partial \mathrm{v}}{\partial p}\right] + p\,\mathfrak{n}\frac{\partial \mathrm{v}}{\partial \theta}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}$$

$$(\partial \mathrm{v}) = \left[\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} + \mathfrak{n}\frac{\partial \mathrm{v}}{\partial p}\right]\left[\mu_k\frac{l_{m_h}}{\theta} - \mu_h\frac{l_{m_k}}{\theta}\right] + \left[\mathfrak{n}\frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}\,(\mathrm{m_1} + \cdots + \mathrm{m_n})\frac{c_p}{\theta}\right]\left[\mu_k\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} - \mu_h\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right]$$

$$(\partial \chi) = \mathfrak{n}\left(p\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right)\left[\mu_k\frac{l_{m_h}}{\theta} - \mu_h\frac{l_{m_k}}{\theta}\right] + p\,\mathfrak{n}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\left(\mu_k\frac{\partial \mathrm{v}}{\partial \theta} + \mathrm{v}\frac{l_{m_k}}{\theta}\right) - \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left(\mu_h\frac{\partial \mathrm{v}}{\partial \theta} + \mathrm{v}\frac{l_{m_h}}{\theta}\right)\right]$$

$$(\partial \psi) = p\,\mathfrak{n}\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_h\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_k\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right] + p\,\mathfrak{n}\frac{\partial \mathrm{v}}{\partial p}\left[\mu_h\frac{l_{m_k}}{\theta} - \mu_k\frac{l_{m_h}}{\theta}\right] + \mathrm{v}\,\mathfrak{n}\left[\frac{l_{m_k}}{\theta}\left(\mu_h - p\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right) - \frac{l_{m_h}}{\theta}\left(\mu_k - p\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)\right]$$

$$(d\mathrm{W}) = p\left[\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} + \mathfrak{n}\frac{\partial \mathrm{v}}{\partial p}\right]\left[\mu_h\frac{l_{m_k}}{\theta} - \mu_k\frac{l_{m_h}}{\theta}\right] + p\left[\mathfrak{n}\frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}\,(\mathrm{m_1} + \cdots + \mathrm{m_n})\frac{c_p}{\theta}\right]\left[\mu_h\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_k\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right]$$

$$(d\mathrm{Q}) = 0$$

Group 84

$\mathrm{m_g}$, ε, $\mathfrak{n}$, χ constant.

$$(\partial \theta) = p\frac{\partial \mathrm{v}}{\partial \theta}\left(\mu_k\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} - \mu_h\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right) + p\frac{\partial \mathrm{v}}{\partial p}\left(\mu_k\frac{l_{m_h}}{\theta} - \mu_h\frac{l_{m_k}}{\theta}\right) + \mathrm{v}\left[\frac{l_{m_h}}{\theta}\left(\mu_k - p\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right) - \frac{l_{m_k}}{\theta}\left(\mu_h - p\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)\right]$$

$$(\partial p) = p\frac{\partial \mathrm{v}}{\partial \theta}\left(\mu_h\frac{l_{m_k}}{\theta} - \mu_k\frac{l_{m_h}}{\theta}\right) + p\,(\mathrm{m_1} + \cdots + \mathrm{m_n})\frac{c_p}{\theta}\left[\mu_k\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} - \mu_h\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right]$$

Group 84 (Con.)

$$(\partial \mathrm{m_h}) = -\frac{\partial \mathrm{v}}{\partial \theta}\left[p\,\frac{\partial \mathrm{v}}{\partial \theta}\,\mu_{\mathrm{k}} + p\,\mathrm{v}\,\frac{l_{m_\mathrm{k}}}{\theta}\right] - (\mathrm{m_1} + \cdots + \mathrm{m_n})\,\frac{c_p}{\theta}\left[\mu_{\mathrm{k}}\left(p\,\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right) - p\,\mathrm{v}\,\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right]$$

$$(\partial \mathrm{m_k}) = p\,\frac{\partial \mathrm{v}}{\partial \theta}\left[\frac{\partial \mathrm{v}}{\partial \theta}\,\mu_{\mathrm{h}} + \mathrm{v}\,\frac{l_{m_\mathrm{h}}}{\theta}\right] + (\mathrm{m_1} + \cdots + \mathrm{m_n})\,\frac{c_p}{\theta}\left[\mu_{\mathrm{h}}\left(p\,\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right) - p\,\mathrm{v}\,\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right]$$

$$(\partial \mathrm{v}) = \mathrm{v}\,\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_{\mathrm{k}}\,\frac{l_{m_\mathrm{h}}}{\theta} - \mu_{\mathrm{h}}\,\frac{l_{m_\mathrm{k}}}{\theta}\right] + \mathrm{v}\,(\mathrm{m_1} + \cdots + \mathrm{m_n})\,\frac{c_p}{\theta}\left[\mu_{\mathrm{h}}\,\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_{\mathrm{k}}\,\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right]$$

$$(\partial \zeta) = p\,\mathfrak{n}\,\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_{\mathrm{h}}\,\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_{\mathrm{k}}\,\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right] + p\,\mathfrak{n}\,\frac{\partial \mathrm{v}}{\partial p}\left(\mu_{\mathrm{h}}\,\frac{l_{m_\mathrm{k}}}{\theta} - \mu_{\mathrm{k}}\,\frac{l_{m_\mathrm{h}}}{\theta}\right) + \mathfrak{n}\,\mathrm{v}\left[\frac{l_{m_\mathrm{k}}}{\theta}\left(\mu_{\mathrm{h}} - p\,\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right) - \frac{l_{m_\mathrm{h}}}{\theta}\left(\mu_{\mathrm{k}} - p\,\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)\right]$$

$$(\partial \psi) = p\,\mathfrak{n}\,\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_{\mathrm{h}}\,\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_{\mathrm{k}}\,\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right] + p\,\mathfrak{n}\,\frac{\partial \mathrm{v}}{\partial p}\left(\mu_{\mathrm{h}}\,\frac{l_{m_\mathrm{k}}}{\theta} - \mu_{\mathrm{k}}\,\frac{l_{m_\mathrm{h}}}{\theta}\right) + \mathfrak{n}\,\mathrm{v}\left[\frac{l_{m_\mathrm{k}}}{\theta}\left(\mu_{\mathrm{h}} - p\,\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right) - \frac{l_{m_\mathrm{h}}}{\theta}\left(\mu_{\mathrm{k}} - p\,\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)\right]$$

$$(d\mathrm{W}) = p\,\mathrm{v}\,\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_{\mathrm{h}}\,\frac{l_{m_\mathrm{k}}}{\theta} - \mu_{\mathrm{k}}\,\frac{l_{m_\mathrm{h}}}{\theta}\right] + p\,\mathrm{v}\,(\mathrm{m_1} + \cdots + \mathrm{m_n})\,\frac{c_p}{\theta}\left[\mu_{\mathrm{k}}\,\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}} - \mu_{\mathrm{h}}\,\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right]$$

$$(d\mathrm{Q}) = 0$$

Group 85

$\mathrm{m_g}$, ε, $\mathfrak{n}$, ψ constant.

θ is constant. Same as *Group 48.*

Group 86

$\mathbf{m}_g$, ε, ζ, χ constant.

$$(\partial\theta) = \mathbf{v}\left[\mu_h\, l_{m_k} - \mu_k\, l_{m_h}\right] + p\,\mathbf{v}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} l_{m_h} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} l_{m_k}\right] + p\,\theta\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + p\frac{\partial \mathbf{v}}{\partial p}\left[\mu_h\, l_{m_k} - \mu_k\, l_{m_h}\right]$$

$$(\partial p) = p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k\, l_{m_h} - \mu_h\, l_{m_k}\right] + p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left[\mu_h\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathfrak{n}\,p\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k})\right]$$

$$(\partial \mathbf{m}_h) = p\,\theta\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{v}\frac{l_{m_k}}{\theta} + \mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left[\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right)\mu_k - p\,\mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] + \mathfrak{n}\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right)\left[\mu_k + l_{m_k}\right] - p\,\mathbf{v}\,\mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}$$

$$(\partial \mathbf{m}_k) = -\,p\,\theta\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{v}\frac{l_{m_h}}{\theta} + \mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left[\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right)\mu_h - p\,\mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] - \mathfrak{n}\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right)\left[\mu_h + l_{m_h}\right] + p\,\mathbf{v}\,\mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}$$

$$(\partial \mathbf{v}) = \mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h\, l_{m_k} - \mu_k\, l_{m_h}\right] + \mathbf{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left[\mu_k\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \mu_h\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] + \mathbf{v}\,\mathfrak{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}(\mu_k + l_{m_k}) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}(\mu_h + l_{m_h})\right]$$

$$(\partial \mathfrak{n}) = \mathfrak{n}\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right)\left[\mu_k\frac{l_{m_h}}{\theta} - \mu_h\frac{l_{m_k}}{\theta}\right] + p\,\mathfrak{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left(\mu_k\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{v}\frac{l_{m_k}}{\theta}\right) - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left(\mu_h\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{v}\frac{l_{m_h}}{\theta}\right)\right]$$

ψ is constant.

Group 86 (Con.)

$$(\mathrm{dW}) = p\,\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_{\mathrm{k}}\, l_{m_{\mathrm{h}}} - \mu_{\mathrm{h}}\, l_{m_{\mathrm{k}}}\right] + p\,\mathrm{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_{\mathrm{n}})\, c_p\left[\mu_{\mathrm{h}}\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}} - \mu_{\mathrm{k}}\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right] + p\,\mathrm{v}\,\mathbf{n}\left[\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}(\mu_{\mathrm{h}} + l_{m_{\mathrm{h}}}) - \frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{h}}}(\mu_{\mathrm{k}} + l_{m_{\mathrm{k}}})\right]$$

$$(\mathrm{dQ}) = \mathbf{n}\left(p\frac{\partial \mathrm{v}}{\partial p} + \mathrm{v}\right)\left[\mu_{\mathrm{k}}\, l_{m_{\mathrm{h}}} - \mu_{\mathrm{h}}\, l_{m_{\mathrm{k}}}\right] + \theta\, p\,\mathbf{n}\left[\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{h}}}\left(\mu_{\mathrm{k}}\frac{\partial \mathrm{v}}{\partial \theta} + \mathrm{v}\frac{l_{m_{\mathrm{k}}}}{\theta}\right) - \frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}\left(\mu_{\mathrm{h}}\frac{\partial \mathrm{v}}{\partial \theta} + \mathrm{v}\frac{l_{m_{\mathrm{h}}}}{\theta}\right)\right]$$

Group 87

$\mathbf{m}_g$, ε, ζ, ψ constant.

χ is constant. Same as *Group 86*.

Group 88

$\mathbf{m}_g$, ε, χ, ψ constant.

ζ is constant. Same as *Group 86*.

Group 89

$\mathbf{m}_g$, $\mathbf{n}$, ζ, χ constant.

θ is constant. Same as *Group 52*.

Group 90

$\mathbf{m}_g$, $\mathbf{n}$, ζ, ψ constant.

$$(\partial\theta) = p\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_{\mathrm{h}}\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}} - \mu_{\mathrm{k}}\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right] + p\frac{\partial \mathrm{v}}{\partial p}\left[\mu_{\mathrm{h}}\frac{l_{m_{\mathrm{k}}}}{\theta} - \mu_{\mathrm{k}}\frac{l_{m_{\mathrm{h}}}}{\theta}\right] + \mathrm{v}\left[\frac{l_{m_{\mathrm{k}}}}{\theta}\left(\mu_{\mathrm{h}} - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{h}}}\right) - \frac{l_{m_{\mathrm{h}}}}{\theta}\left(\mu_{\mathrm{k}} - p\frac{\partial \mathrm{v}}{\partial \mathbf{m}_{\mathrm{k}}}\right)\right]$$

Group 90 (Con.)

$$(\partial p) = p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k\frac{l_{m_h}}{\theta} - \mu_h\frac{l_{m_k}}{\theta}\right] + p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_h\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathbf{n}\,p\left[\frac{l_{m_h}}{\theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \frac{l_{m_k}}{\theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right]$$

$$(\partial \mathbf{m}_h) = p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{v}\frac{l_{m_k}}{\theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right)\mu_k - p\,\mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] + \mathbf{n}\frac{l_{m_k}}{\theta}\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right)$$

$$(\partial \mathbf{m}_k) = -p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{v}\frac{l_{m_h}}{\theta} + \mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right)\mu_h - p\,\mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] - \mathbf{n}\frac{l_{m_h}}{\theta}\left(p\frac{\partial \mathbf{v}}{\partial p} + \mathbf{v}\right)$$

$$(\partial \mathbf{v}) = \mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h\frac{l_{m_k}}{\theta} - \mu_k\frac{l_{m_h}}{\theta}\right] + \mathbf{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_k\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \mu_h\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] + \mathbf{n}\,\mathbf{v}\left[\frac{l_{m_k}}{\theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \frac{l_{m_h}}{\theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right]$$

$$(\partial \boldsymbol{\varepsilon}) = p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial p}\left[\mu_h\frac{l_{m_k}}{\theta} - \mu_k\frac{l_{m_h}}{\theta}\right] + \mathbf{v}\,\mathbf{n}\left[\frac{l_{m_k}}{\theta}\left(\mu_h - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right) - \frac{l_{m_h}}{\theta}\left(\mu_k - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)\right]$$

$$(\partial \chi) = p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial p}\left[\mu_h\frac{l_{m_k}}{\theta} - \mu_k\frac{l_{m_h}}{\theta}\right] + \mathbf{v}\,\mathbf{n}\left[\frac{l_{m_k}}{\theta}\left(\mu_h - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right) - \frac{l_{m_h}}{\theta}\left(\mu_k - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)\right]$$

$$(d\mathbf{W}) = p\,\mathbf{v}\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k\frac{l_{m_h}}{\theta} - \mu_h\frac{l_{m_k}}{\theta}\right] - p\,\mathbf{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\frac{c_p}{\theta}\left[\mu_k\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \mu_h\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] + p\,\mathbf{n}\,\mathbf{v}\left[\frac{l_{m_h}}{\theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \frac{l_{m_k}}{\theta}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right]$$

$$(d\mathbf{Q}) = 0$$

Group 91

m_g, $\mathfrak{n}$, χ, ψ constant.

$$(\partial\theta) = p\frac{\partial v}{\partial\theta}\left[\mu_h\frac{\partial v}{\partial m_k} - \mu_k\frac{\partial v}{\partial m_h}\right] + p\frac{\partial v}{\partial p}\left[\mu_h\frac{l_{m_k}}{\theta} - \mu_k\frac{l_{m_h}}{\theta}\right] + v\left[\frac{l_{m_k}}{\theta}\left(\mu_h - p\frac{\partial v}{\partial m_h}\right) - \frac{l_{m_h}}{\theta}\left(\mu_k - p\frac{\partial v}{\partial m_k}\right)\right]$$

$$(\partial p) = \left(\mathfrak{n} + p\frac{\partial v}{\partial\theta}\right)\left[\mu_k\frac{l_{m_h}}{\theta} - \mu_h\frac{l_{m_k}}{\theta}\right] + p\,(m_1 + \cdots + m_n)\frac{c_p}{\theta}\left[\mu_h\frac{\partial v}{\partial m_k} - \mu_k\frac{\partial v}{\partial m_h}\right]$$

$$(\partial m_h) = \frac{\partial v}{\partial\theta}\left[\left(p\frac{\partial v}{\partial\theta} + \mathfrak{n}\right)\mu_k + p\,v\frac{l_{m_k}}{\theta}\right] + (m_1 + \cdots + m_n)\frac{c_p}{\theta}\left[\left(p\frac{\partial v}{\partial p} + v\right)\mu_k - p\,v\frac{\partial v}{\partial m_k}\right] + \mathfrak{n}\,v\frac{l_{m_k}}{\theta}$$

$$(\partial m_k) = -\frac{\partial v}{\partial\theta}\left[\left(p\frac{\partial v}{\partial\theta} + \mathfrak{n}\right)\mu_h + p\,v\frac{l_{m_h}}{\theta}\right] - (m_1 + \cdots + m_n)\frac{c_p}{\theta}\left[\left(p\frac{\partial v}{\partial p} + v\right)\mu_h - p\,v\frac{\partial v}{\partial m_h}\right] - \mathfrak{n}\,v\frac{l_{m_h}}{\theta}$$

$$(\partial v) = \left[v\frac{\partial v}{\partial\theta} - \mathfrak{n}\frac{\partial v}{\partial p}\right]\left[\mu_h\frac{l_{m_k}}{\theta} - \mu_k\frac{l_{m_h}}{\theta}\right] - \left[v\,(m_1 + \cdots + m_n)\frac{c_p}{\theta} + \mathfrak{n}\frac{\partial v}{\partial\theta}\right]\left[\mu_h\frac{\partial v}{\partial m_k} - \mu_k\frac{\partial v}{\partial m_h}\right] + \mathfrak{n}\,v\left[\frac{\partial v}{\partial m_h}\frac{l_{m_k}}{\theta} - \frac{\partial v}{\partial m_k}\frac{l_{m_h}}{\theta}\right]$$

$$(\partial\varepsilon) = p\,\mathfrak{n}\frac{\partial v}{\partial\theta}\left[\mu_h\frac{\partial v}{\partial m_k} - \mu_k\frac{\partial v}{\partial m_h}\right] + p\,\mathfrak{n}\frac{\partial v}{\partial p}\left[\mu_h\frac{l_{m_k}}{\theta} - \mu_k\frac{l_{m_h}}{\theta}\right] + \mathfrak{n}\,v\left[\frac{l_{m_k}}{\theta}\left(\mu_h - p\frac{\partial v}{\partial m_h}\right) - \frac{l_{m_h}}{\theta}\left(\mu_k - p\frac{\partial v}{\partial m_k}\right)\right]$$

$$(\partial\zeta) = p\,\mathfrak{n}\frac{\partial v}{\partial\theta}\left[\mu_k\frac{\partial v}{\partial m_h} - \mu_h\frac{\partial v}{\partial m_k}\right] + p\,\mathfrak{n}\frac{\partial v}{\partial p}\left[\mu_k\frac{l_{m_h}}{\theta} - \mu_h\frac{l_{m_k}}{\theta}\right] + \mathfrak{n}\,v\left[\frac{l_{m_h}}{\theta}\left(\mu_k - p\frac{\partial v}{\partial m_k}\right) - \frac{l_{m_k}}{\theta}\left(\mu_h - p\frac{\partial v}{\partial m_h}\right)\right]$$

Group 91 (Con.)

$$(\mathrm{dW}) = p\left[\mathbf{v}\frac{\partial\mathbf{v}}{\partial\theta} - \mathbf{n}\frac{\partial\mathbf{v}}{\partial p}\right]\left[\mu_{\mathrm{k}}\frac{l_{m_\mathrm{h}}}{\theta} - \mu_{\mathrm{h}}\frac{l_{m_\mathrm{k}}}{\theta}\right] + p\left[\mathbf{v}\,(\mathbf{m}_1 + \cdots + \mathbf{m}_\mathrm{n})\frac{c_p}{\theta} + \mathbf{n}\frac{\partial\mathbf{v}}{\partial\theta}\right]\left[\mu_{\mathrm{h}}\frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{k}} - \mu_{\mathrm{k}}\frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{h}}\right] + p\,\mathbf{n}\,\mathbf{v}\left[\frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{k}}\frac{l_{m_\mathrm{h}}}{\theta} - \frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{h}}\frac{l_{m_\mathrm{k}}}{\theta}\right]$$

$$(\mathrm{d}\mathbf{Q}) = 0$$

Group 92

$\mathbf{m}_\mathrm{g}$, ζ, χ, ψ constant.

ε is constant. Same as *Group 86.*

Group 93

θ, p, $\mathbf{m}_\mathrm{b}$, $\mathbf{v}$, ε constant.

$$(\partial\mathbf{m}_\mathrm{y}) = \frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{k}}(\mu_\mathrm{h} + l_{m_\mathrm{h}}) - \frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{h}}(\mu_\mathrm{k} + l_{m_\mathrm{k}})$$

$$(\partial\mathbf{m}_\mathrm{h}) = \frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{y}}(\mu_\mathrm{k} + l_{m_\mathrm{k}}) - \frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{k}}(\mu_\mathrm{y} + l_{m_\mathrm{y}})$$

$$(\partial\mathbf{m}_\mathrm{k}) = -\frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{y}}(\mu_\mathrm{h} + l_{m_\mathrm{h}}) + \frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{h}}(\mu_\mathrm{y} + l_{m_\mathrm{y}})$$

$$(\partial\mathbf{n}) = \frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{k}}\left[\frac{l_{m_\mathrm{y}}}{\theta}\mu_\mathrm{h} - \frac{l_{m_\mathrm{h}}}{\theta}\mu_\mathrm{y}\right] - \frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{h}}\left[\frac{l_{m_\mathrm{y}}}{\theta}\mu_\mathrm{k} - \frac{l_{m_\mathrm{k}}}{\theta}\mu_\mathrm{y}\right] - \frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{y}}\left[\frac{l_{m_\mathrm{k}}}{\theta}\mu_\mathrm{h} - \frac{l_{m_\mathrm{h}}}{\theta}\mu_\mathrm{k}\right]$$

$$(\partial\zeta) = \frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{k}}\left[l_{m_\mathrm{h}}\mu_\mathrm{y} - l_{m_\mathrm{y}}\mu_\mathrm{h}\right] - \frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{h}}\left[l_{m_\mathrm{k}}\mu_\mathrm{y} - l_{m_\mathrm{y}}\mu_\mathrm{k}\right] - \frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{y}}\left[l_{m_\mathrm{h}}\mu_\mathrm{k} - l_{m_\mathrm{k}}\mu_\mathrm{h}\right]$$

χ is constant.

$$(\partial\psi) = \frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{k}}\left[l_{m_\mathrm{h}}\mu_\mathrm{y} - l_{m_\mathrm{y}}\mu_\mathrm{h}\right] - \frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{h}}\left[l_{m_\mathrm{k}}\mu_\mathrm{y} - l_{m_\mathrm{y}}\mu_\mathrm{k}\right] - \frac{\partial\mathbf{v}}{\partial\mathbf{m}_\mathrm{y}}\left[l_{m_\mathrm{h}}\mu_\mathrm{k} - l_{m_\mathrm{k}}\mu_\mathrm{h}\right]$$

Group 93 (Con.)

$$(dW) = 0$$

$$(dQ) = \frac{\partial v}{\partial m_k}\left[l_{m_y}\mu_h - l_{m_h}\mu_y\right] - \frac{\partial v}{\partial m_h}\left[l_{m_y}\mu_k - l_{m_k}\mu_y\right] - \frac{\partial v}{\partial m_y}\left[l_{m_k}\mu_h - l_{m_h}\mu_k\right]$$

Group 94

θ, p, m_b, v, n constant.

$$(\partial m_y) = \frac{\partial v}{\partial m_k}\frac{l_{m_h}}{\theta} - \frac{\partial v}{\partial m_h}\frac{l_{m_k}}{\theta}$$

$$(\partial m_h) = \frac{\partial v}{\partial m_y}\frac{l_{m_k}}{\theta} - \frac{\partial v}{\partial m_k}\frac{l_{m_y}}{\theta}$$

$$(\partial m_k) = -\frac{\partial v}{\partial m_y}\frac{l_{m_h}}{\theta} + \frac{\partial v}{\partial m_h}\frac{l_{m_y}}{\theta}$$

$$(\partial \varepsilon) = \frac{\partial v}{\partial m_k}\left[\frac{l_{m_h}}{\theta}\mu_y - \frac{l_{m_y}}{\theta}\mu_h\right] - \frac{\partial v}{\partial m_h}\left[\frac{l_{m_k}}{\theta}\mu_y - \frac{l_{m_y}}{\theta}\mu_k\right] - \frac{\partial v}{\partial m_y}\left[\frac{l_{m_h}}{\theta}\mu_k - \frac{l_{m_k}}{\theta}\mu_h\right]$$

$$(\partial \zeta) = \frac{\partial v}{\partial m_k}\left[\frac{l_{m_h}}{\theta}\mu_y - \frac{l_{m_y}}{\theta}\mu_h\right] - \frac{\partial v}{\partial m_h}\left[\frac{l_{m_k}}{\theta}\mu_y - \frac{l_{m_y}}{\theta}\mu_k\right] - \frac{\partial v}{\partial m_y}\left[\frac{l_{m_h}}{\theta}\mu_k - \frac{l_{m_k}}{\theta}\mu_h\right]$$

$$(\partial \chi) = \frac{\partial v}{\partial m_k}\left[\frac{l_{m_h}}{\theta}\mu_y - \frac{l_{m_y}}{\theta}\mu_h\right] - \frac{\partial v}{\partial m_h}\left[\frac{l_{m_k}}{\theta}\mu_y - \frac{l_{m_y}}{\theta}\mu_k\right] - \frac{\partial v}{\partial m_y}\left[\frac{l_{m_h}}{\theta}\mu_k - \frac{l_{m_k}}{\theta}\mu_h\right]$$

$$(\partial \psi) = \frac{\partial v}{\partial m_k}\left[\frac{l_{m_h}}{\theta}\mu_y - \frac{l_{m_y}}{\theta}\mu_h\right] - \frac{\partial v}{\partial m_h}\left[\frac{l_{m_k}}{\theta}\mu_y - \frac{l_{m_y}}{\theta}\mu_k\right] - \frac{\partial v}{\partial m_y}\left[\frac{l_{m_h}}{\theta}\mu_k - \frac{l_{m_k}}{\theta}\mu_h\right]$$

$$(dW) = 0$$

$$(dQ) = 0$$

Group 95

θ, p, $\mathbf{m}_b$, $\mathbf{v}$, ζ constant.

$$(\partial \mathbf{m}_y) = \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} \mu_h - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} \mu_k$$

$$(\partial \mathbf{m}_h) = \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y} \mu_k - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} \mu_y$$

$$(\partial \mathbf{m}_k) = - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y} \mu_h + \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} \mu_y$$

$$(\partial \boldsymbol{\varepsilon}) = \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[l_{m_y} \mu_h - l_{m_h} \mu_y\right] - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_y} \mu_k - l_{m_k} \mu_y\right] - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_k} \mu_h - l_{m_h} \mu_k\right]$$

$$(\partial \boldsymbol{\eta}) = \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[\frac{l_{m_y}}{\theta} \mu_h - \frac{l_{m_h}}{\theta} \mu_y\right] - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[\frac{l_{m_y}}{\theta} \mu_k - \frac{l_{m_k}}{\theta} \mu_y\right] - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[\frac{l_{m_k}}{\theta} \mu_h - \frac{l_{m_h}}{\theta} \mu_k\right]$$

$$(\partial \boldsymbol{\chi}) = \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[l_{m_y} \mu_h - l_{m_h} \mu_y\right] - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_y} \mu_k - l_{m_k} \mu_y\right] - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_k} \mu_h - l_{m_h} \mu_k\right]$$

$\boldsymbol{\psi}$ is constant.

$$(d\mathbf{W}) = 0$$

$$(d\mathbf{Q}) = \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[l_{m_y} \mu_h - l_{m_h} \mu_y\right] - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_y} \mu_k - l_{m_k} \mu_y\right] - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_k} \mu_h - l_{m_h} \mu_k\right]$$

Group 96

θ, p, $\mathbf{m}_b$, $\mathbf{v}$, $\boldsymbol{\chi}$ constant.

$\boldsymbol{\varepsilon}$ is constant. Same as *Group 93*.

Group 97

θ, p, $\mathbf{m}_b$, $\mathbf{v}$, ψ constant.

ζ is constant. Same as *Group 95*.

Group 98

θ, p, $\mathbf{m}_b$, ε, η constant.

$$(\partial \mathbf{m}_y) = l_{m_b}\left(\mu_k - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right) - l_{m_k}\left(\mu_h - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)$$

$$(\partial \mathbf{m}_h) = l_{m_k}\left(\mu_y - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\right) - l_{m_y}\left(\mu_k - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)$$

$$(\partial \mathbf{m}_k) = -\, l_{m_h}\left(\mu_y - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\right) + l_{m_y}\left(\mu_h - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)$$

$$(\partial \mathbf{v}) = \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[l_{m_y}\mu_h - l_{m_h}\mu_y\right] - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_y}\mu_k - l_{m_k}\mu_y\right] - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_k}\mu_h - l_{m_h}\mu_k\right]$$

$$(\partial \zeta) = p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[l_{m_y}\mu_h - l_{m_h}\mu_y\right] - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_y}\mu_k - l_{m_k}\mu_y\right] - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_k}\mu_h - l_{m_h}\mu_k\right]$$

$$(\partial \chi) = p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[l_{m_y}\mu_h - l_{m_h}\mu_y\right] - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_y}\mu_k - l_{m_k}\mu_y\right] - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_k}\mu_h - l_{m_h}\mu_k\right]$$

ψ is constant.

$$(d\mathbf{W}) = p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[l_{m_h}\mu_y - l_{m_y}\mu_h\right] - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_k}\mu_y - l_{m_y}\mu_k\right] - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_h}\mu_k - l_{m_k}\mu_h\right]$$

$$(d\mathbf{Q}) = 0$$

Group 99

θ, p, $\mathbf{m}_b$, $\mathfrak{r}$, ζ constant.

$$(\partial \mathbf{m}_y) = \mu_h \left[l_{m_k} - p \frac{\partial v}{\partial \mathbf{m}_k} \right] - \mu_k \left[l_{m_h} - p \frac{\partial v}{\partial \mathbf{m}_h} \right]$$

$$(\partial \mathbf{m}_h) = \mu_k \left[l_{m_y} - p \frac{\partial v}{\partial \mathbf{m}_y} \right] - \mu_y \left[l_{m_k} - p \frac{\partial v}{\partial \mathbf{m}_k} \right]$$

$$(\partial \mathbf{m}_k) = - \mu_h \left[l_{m_y} - p \frac{\partial v}{\partial \mathbf{m}_y} \right] + \mu_y \left[l_{m_h} - p \frac{\partial v}{\partial \mathbf{m}_h} \right]$$

$$(\partial v) = \frac{\partial v}{\partial \mathbf{m}_k} \left[l_{m_h} \mu_y - l_{m_y} \mu_h \right] - \frac{\partial v}{\partial \mathbf{m}_h} \left[l_{m_k} \mu_y - l_{m_y} \mu_k \right] -$$
$$\frac{\partial v}{\partial \mathbf{m}_y} \left[l_{m_h} \mu_k - l_{m_k} \mu_h \right]$$

$$(\partial \mathfrak{n}) = p \frac{\partial v}{\partial \mathbf{m}_k} \left[\frac{l_{m_h}}{\theta} \mu_y - \frac{l_{m_y}}{\theta} \mu_h \right] - p \frac{\partial v}{\partial \mathbf{m}_h} \left[\frac{l_{m_k}}{\theta} \mu_y - \frac{l_{m_y}}{\theta} \mu_k \right] -$$
$$p \frac{\partial v}{\partial \mathbf{m}_y} \left[\frac{l_{m_h}}{\theta} \mu_k - \frac{l_{m_k}}{\theta} \mu_h \right]$$

$$(\partial \chi) = p \frac{\partial v}{\partial \mathbf{m}_k} \left[l_{m_h} \mu_y - l_{m_y} \mu_h \right] - p \frac{\partial v}{\partial \mathbf{m}_h} \left[l_{m_k} \mu_y - l_{m_y} \mu_k \right] -$$
$$p \frac{\partial v}{\partial \mathbf{m}_y} \left[l_{m_h} \mu_k - l_{m_k} \mu_h \right]$$

$$(\partial \psi) = p \frac{\partial v}{\partial \mathbf{m}_k} \left[l_{m_y} \mu_h - l_{m_h} \mu_y \right] - p \frac{\partial v}{\partial \mathbf{m}_h} \left[l_{m_y} \mu_k - l_{m_k} \mu_y \right] -$$
$$p \frac{\partial v}{\partial \mathbf{m}_y} \left[l_{m_k} \mu_h - l_{m_h} \mu_k \right]$$

$$(dW) = p \frac{\partial v}{\partial \mathbf{m}_k} \left[l_{m_y} \mu_h - l_{m_h} \mu_y \right] - p \frac{\partial v}{\partial \mathbf{m}_h} \left[l_{m_y} \mu_k - l_{m_k} \mu_y \right] -$$
$$p \frac{\partial v}{\partial \mathbf{m}_y} \left[l_{m_k} \mu_h - l_{m_h} \mu_k \right]$$

$$(dQ) = p \frac{\partial v}{\partial \mathbf{m}_k} \left[l_{m_h} \mu_y - l_{m_y} \mu_h \right] - p \frac{\partial v}{\partial \mathbf{m}_h} \left[l_{m_k} \mu_y - l_{m_y} \mu_k \right] -$$
$$p \frac{\partial v}{\partial \mathbf{m}_y} \left[l_{m_h} \mu_k - l_{m_k} \mu_h \right]$$

Group 100

θ, p, m_b, ε, χ constant.

$\mathbf{v}$ is constant. Same as *Group 93*.

Group 101

θ, p, m_b, ε, ψ constant.

$\mathbf{n}$ is constant. Same as *Group 98*.

Group 102

θ, p, m_b, $\mathbf{n}$, ζ constant.

$$(\partial m_y) = \mu_h l_{m_k} - \mu_k l_{m_h}$$

$$(\partial m_h) = \mu_k l_{m_y} - \mu_y l_{m_k}$$

$$(\partial m_k) = -\mu_h l_{m_y} + \mu_y l_{m_h}$$

$$(\partial v) = \frac{\partial v}{\partial m_k}\left[l_{m_h}\mu_y - l_{m_y}\mu_h\right] - \frac{\partial v}{\partial m_h}\left[l_{m_k}\mu_y - l_{m_y}\mu_k\right] - \frac{\partial v}{\partial m_y}\left[l_{m_h}\mu_k - l_{m_k}\mu_h\right]$$

$$(\partial \varepsilon) = p\frac{\partial v}{\partial m_k}\left[l_{m_y}\mu_h - l_{m_h}\mu_y\right] - p\frac{\partial v}{\partial m_h}\left[l_{m_y}\mu_k - l_{m_k}\mu_y\right] - p\frac{\partial v}{\partial m_y}\left[l_{m_k}\mu_h - l_{m_h}\mu_k\right]$$

χ is constant.

$$(\partial \psi) = p\frac{\partial v}{\partial m_k}\left[l_{m_y}\mu_h - l_{m_h}\mu_y\right] - p\frac{\partial v}{\partial m_h}\left[l_{m_y}\mu_k - l_{m_k}\mu_y\right] - p\frac{\partial v}{\partial m_y}\left[l_{m_k}\mu_h - l_{m_h}\mu_k\right]$$

$$(dW) = p\frac{\partial v}{\partial m_k}\left[l_{m_y}\mu_h - l_{m_h}\mu_y\right] - p\frac{\partial v}{\partial m_h}\left[l_{m_y}\mu_k - l_{m_k}\mu_y\right] - p\frac{\partial v}{\partial m_y}\left[l_{m_k}\mu_h - l_{m_h}\mu_k\right]$$

$$(dQ) = 0$$

Group 103

θ, p, $\mathbf{m}_b$, $\mathfrak{n}$, χ constant.

ζ is constant. Same as *Group 102*.

Group 104

θ, p, $\mathbf{m}_b$, $\mathfrak{n}$, ψ constant.

ε is constant. Same as *Group 98*.

Group 105

θ, p, $\mathbf{m}_b$, ζ, χ constant.

$\mathfrak{n}$ is constant. Same as *Group 102*.

Group 106

θ, p, $\mathbf{m}_b$, ζ, ψ constant.

$\mathbf{v}$ is constant. Same as *Group 95*.

Group 107

θ, p, $\mathbf{m}_b$, χ, ψ constant.

$$(\partial \mathbf{m}_y) = \mu_h l_{m_k} - \mu_k l_{m_h} + p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} (\mu_h + l_{m_h}) - p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} (\mu_h + l_{m_k})$$

$$(\partial \mathbf{m}_h) = \mu_k l_{m_y} - \mu_y l_{m_k} + p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y} (\mu_k + l_{m_k}) - p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} (\mu_y + l_{m_y})$$

$$(\partial \mathbf{m}_k) = - \mu_h l_{m_y} + \mu_y l_{m_h} - p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y} (\mu_h + l_{m_h}) + p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} (\mu_y + l_{m_y})$$

$$(\partial \mathbf{v}) = \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} \left[l_{m_h} \mu_y - l_{m_y} \mu_h \right] - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} \left[l_{m_k} \mu_y - l_{m_y} \mu_k \right] - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y} \left[l_{m_h} \mu_k - l_{m_k} \mu_h \right]$$

$$(\partial \varepsilon) = p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} \left[l_{m_y} \mu_h - l_{m_h} \mu_y \right] - p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} \left[l_{m_y} \mu_k - l_{m_k} \mu_y \right] - p \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y} \left[l_{m_k} \mu_h - l_{m_h} \mu_k \right]$$

Group 107 (Con.)

$$(\partial \mathfrak{n}) = p\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[\frac{l_{m_y}}{\theta}\mu_{\mathrm{h}} - \frac{l_{m_h}}{\theta}\mu_{\mathrm{y}}\right] - p\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\left[\frac{l_{m_y}}{\theta}\mu_{\mathrm{k}} - \frac{l_{m_k}}{\theta}\mu_{\mathrm{y}}\right] - p\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}\left[\frac{l_{m_k}}{\theta}\mu_{\mathrm{h}} - \frac{l_{m_h}}{\theta}\mu_{\mathrm{k}}\right]$$

$$(\partial \zeta) = p\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[l_{m_h}\mu_{\mathrm{y}} - l_{m_y}\mu_{\mathrm{h}}\right] - p\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\left[l_{m_k}\mu_{\mathrm{y}} - l_{m_y}\mu_{\mathrm{k}}\right] - p\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}\left[l_{m_h}\mu_{\mathrm{k}} - l_{m_k}\mu_{\mathrm{h}}\right]$$

$$(d\mathrm{W}) = p\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[l_{m_y}\mu_{\mathrm{h}} - l_{m_h}\mu_{\mathrm{y}}\right] - p\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\left[l_{m_y}\mu_{\mathrm{k}} - l_{m_k}\mu_{\mathrm{y}}\right] - p\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}\left[l_{m_k}\mu_{\mathrm{h}} - l_{m_h}\mu_{\mathrm{k}}\right]$$

$$(d\mathbf{Q}) = p\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[l_{m_y}\mu_{\mathrm{h}} - l_{m_h}\mu_{\mathrm{y}}\right] - p\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\left[l_{m_y}\mu_{\mathrm{k}} - l_{m_k}\mu_{\mathrm{y}}\right] - p\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}\left[l_{m_k}\mu_{\mathrm{h}} - l_{m_h}\mu_{\mathrm{k}}\right]$$

Group 108

θ, $\mathrm{m_b}$, v, $\boldsymbol{\varepsilon}$, $\mathfrak{n}$ constant.

$$(\partial p) = \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[\mu_{\mathrm{h}} l_{m_y} - \mu_{\mathrm{y}} l_{m_h}\right] - \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\left[\mu_{\mathrm{k}} l_{m_y} - \mu_{\mathrm{y}} l_{m_k}\right] - \frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}\left[\mu_{\mathrm{h}} l_{m_k} - \mu_{\mathrm{k}} l_{m_h}\right]$$

$$(\partial \mathrm{m_y}) = \theta\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_{\mathrm{h}}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_{\mathrm{k}}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right] + \frac{\partial \mathrm{v}}{\partial p}\left[\mu_{\mathrm{h}} l_{m_k} - \mu_{\mathrm{k}} l_{m_h}\right]$$

$$(\partial \mathrm{m_h}) = \theta\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_{\mathrm{k}}\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}} - \mu_{\mathrm{y}}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right] + \frac{\partial \mathrm{v}}{\partial p}\left[\mu_{\mathrm{k}} l_{m_y} - \mu_{\mathrm{y}} l_{m_k}\right]$$

$$(\partial \mathrm{m_k}) = -\theta\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_{\mathrm{h}}\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}} - \mu_{\mathrm{y}}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right] - \frac{\partial \mathrm{v}}{\partial p}\left[\mu_{\mathrm{h}} l_{m_y} - \mu_{\mathrm{y}} l_{m_h}\right]$$

$$(\partial \zeta) = \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[\mu_{\mathrm{h}} l_{m_y} - \mu_{\mathrm{y}} l_{m_h}\right] - \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\left[\mu_{\mathrm{k}} l_{m_y} - \mu_{\mathrm{y}} l_{m_k}\right] - \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}\left[\mu_{\mathrm{h}} l_{m_k} - \mu_{\mathrm{k}} l_{m_h}\right]$$

Group 114 (Con.)

$$(\partial\chi) = \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[\mu_\mathrm{h}\, l_{m_\mathrm{y}} - \mu_\mathrm{y}\, l_{m_\mathrm{h}}\right] - \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\left[\mu_\mathrm{k}\, l_{m_\mathrm{y}} - \mu_\mathrm{y}\, l_{m_\mathrm{k}}\right] -$$
$$\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}\left[\mu_\mathrm{h}\, l_{m_\mathrm{k}} - \mu_\mathrm{k}\, l_{m_\mathrm{h}}\right]$$

ψ is constant.

$$(d\mathbf{W}) = 0$$

$$(d\mathbf{Q}) = 0$$

Group 109

θ, $\mathrm{m_b}$, v, ε, ζ constant.

$$(\partial p) = \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[l_{m_\mathrm{y}}\, \mu_\mathrm{h} - l_{m_\mathrm{h}}\, \mu_\mathrm{y}\right] - \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\left[l_{m_\mathrm{y}}\, \mu_\mathrm{k} - l_{m_\mathrm{k}}\, \mu_\mathrm{y}\right] -$$
$$\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}\left[l_{m_\mathrm{k}}\, \mu_\mathrm{h} - l_{m_\mathrm{h}}\, \mu_\mathrm{k}\right]$$

$$(\partial \mathrm{m_y}) = \frac{\partial \mathrm{v}}{\partial p}\left[\mu_\mathrm{h}\, l_{m_\mathrm{k}} - \mu_\mathrm{k}\, l_{m_\mathrm{h}}\right] + \theta\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_\mathrm{h}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_\mathrm{k}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right] +$$
$$\mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}(\mu_\mathrm{h} + l_{m_\mathrm{h}}) - \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}(\mu_\mathrm{k} + l_{m_\mathrm{k}})\right]$$

$$(\partial \mathrm{m_h}) = \frac{\partial \mathrm{v}}{\partial p}\left[\mu_\mathrm{k}\, l_{m_\mathrm{y}} - \mu_\mathrm{y}\, l_{m_\mathrm{k}}\right] + \theta\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_\mathrm{k}\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}} - \mu_\mathrm{y}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right] +$$
$$\mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}(\mu_\mathrm{k} + l_{m_\mathrm{k}}) - \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}(\mu_\mathrm{y} + l_{m_\mathrm{y}})\right]$$

$$(\partial \mathrm{m_k}) = -\frac{\partial \mathrm{v}}{\partial p}\left[\mu_\mathrm{h}\, l_{m_\mathrm{y}} - \mu_\mathrm{y}\, l_{m_\mathrm{h}}\right] - \theta\frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_\mathrm{h}\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}} - \mu_\mathrm{y}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right] -$$
$$\mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}(\mu_\mathrm{h} + l_{m_\mathrm{h}}) - \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}(\mu_\mathrm{y} + l_{m_\mathrm{y}})\right]$$

$$(\partial \mathbf{n}) = \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[\mu_\mathrm{h}\frac{l_{m_\mathrm{y}}}{\theta} - \mu_\mathrm{y}\frac{l_{m_\mathrm{h}}}{\theta}\right] - \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\left[\mu_\mathrm{k}\frac{l_{m_\mathrm{y}}}{\theta} - \mu_\mathrm{y}\frac{l_{m_\mathrm{k}}}{\theta}\right] -$$
$$\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}\left[\mu_\mathrm{h}\frac{l_{m_\mathrm{k}}}{\theta} - \mu_\mathrm{k}\frac{l_{m_\mathrm{h}}}{\theta}\right]$$

Group 109 (Con.)

$$(\partial\chi) = \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[l_{m_y}\mu_h - l_{m_h}\mu_y\right] - \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\left[l_{m_y}\mu_k - l_{m_k}\mu_y\right] - \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}\left[l_{m_k}\mu_h - l_{m_h}\mu_k\right]$$

$$(\partial\psi) = \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[l_{m_h}\mu_y - l_{m_y}\mu_h\right] - \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\left[l_{m_k}\mu_y - l_{m_y}\mu_k\right] + \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}\left[l_{m_k}\mu_h - l_{m_h}\mu_k\right]$$

$$(dW) = 0$$

$$(dQ) = \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[\mu_h l_{m_y} - \mu_y l_{m_h}\right] - \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\left[\mu_k l_{m_y} - \mu_y l_{m_k}\right] - \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}\left[\mu_h l_{m_k} - \mu_k l_{m_h}\right]$$

Group 110

θ, $\mathrm{m_b}$, v, ε, χ constant.

p is constant. Same as *Group 93.*

Group 111

θ, $\mathrm{m_b}$, v, ε, ψ constant.

$\mathfrak{n}$ is constant. Same as *Group 108.*

Group 112

θ, $\mathrm{m_b}$, v, $\mathfrak{n}$, ζ constant.

$$(\partial p) = \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[\frac{l_{m_y}}{\theta}\mu_h - \frac{l_{m_h}}{\theta}\mu_y\right] - \frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\left[\frac{l_{m_y}}{\theta}\mu_k - \frac{l_{m_k}}{\theta}\mu_y\right] - \frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}\left[\frac{l_{m_k}}{\theta}\mu_h - \frac{l_{m_h}}{\theta}\mu_k\right]$$

$$(\partial \mathrm{m_y}) = \frac{\partial \mathrm{v}}{\partial \theta}\left[\mu_h\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_k\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right] + \frac{\partial \mathrm{v}}{\partial p}\left[\mu_h\frac{l_{m_k}}{\theta} - \mu_k\frac{l_{m_h}}{\theta}\right] - \mathrm{v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\frac{l_{m_k}}{\theta} - \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\frac{l_{m_h}}{\theta}\right]$$

Group 112 (Con.)

$$(\partial \mathbf{m}_h) = \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y} - \mu_y \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] + \frac{\partial \mathbf{v}}{\partial p}\left[\mu_k \frac{l_{m_y}}{\theta} - \mu_y \frac{l_{m_k}}{\theta}\right] - \mathbf{v}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\frac{l_{m_y}}{\theta} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\frac{l_{m_k}}{\theta}\right]$$

$$(\partial \mathbf{m}_k) = -\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y} - \mu_y \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] - \frac{\partial \mathbf{v}}{\partial p}\left[\mu_h \frac{l_{m_y}}{\theta} - \mu_y \frac{l_{m_h}}{\theta}\right] + \mathbf{v}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\frac{l_{m_y}}{\theta} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\frac{l_{m_h}}{\theta}\right]$$

$$(\partial \varepsilon) = \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[\frac{l_{m_h}}{\theta}\mu_y - \frac{l_{m_y}}{\theta}\mu_h\right] - \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[\frac{l_{m_k}}{\theta}\mu_y - \frac{l_{m_y}}{\theta}\mu_k\right] - \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[\frac{l_{m_h}}{\theta}\mu_k - \frac{l_{m_k}}{\theta}\mu_h\right]$$

χ is constant.

$$(\partial \psi) = \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[\frac{l_{m_h}}{\theta}\mu_y - \frac{l_{m_y}}{\theta}\mu_h\right] - \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[\frac{l_{m_k}}{\theta}\mu_y - \frac{l_{m_y}}{\theta}\mu_k\right] - \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[\frac{l_{m_h}}{\theta}\mu_k - \frac{l_{m_k}}{\theta}\mu_h\right]$$

$$(d\mathbf{W}) = 0$$

$$(d\mathbf{Q}) = 0$$

Group 113

θ, $\mathbf{m}_b$, $\mathbf{v}$, $\mathfrak{n}$, χ constant.

ζ is constant. Same as *Group 112.*

Group 114

θ, $\mathbf{m}_b$, $\mathbf{v}$, $\mathfrak{n}$, ψ constant.

ε is constant. Same as *Group 108.*

Group 115

θ, $\mathbf{m}_b$, $\mathbf{v}$, ζ, χ constant.

$\mathfrak{n}$ is constant. Same as *Group 112.*

Group 116

θ, m_b, v, ζ, ψ constant.

p is constant. Same as *Group 95.*

Group 117

θ, m_b, v, χ, ψ constant.

$$(\partial p) = \frac{\partial v}{\partial m_k}\left[l_{m_y}\mu_h - l_{m_h}\mu_y\right] - \frac{\partial v}{\partial m_h}\left[l_{m_y}\mu_k - l_{m_k}\mu_y\right] - \frac{\partial v}{\partial m_y}\left[l_{m_k}\mu_h - l_{m_h}\mu_k\right]$$

$$(\partial m_y) = \left[\theta\frac{\partial v}{\partial \theta} - v\right]\left[\mu_h\frac{\partial v}{\partial m_k} - \mu_k\frac{\partial v}{\partial m_h}\right] + \frac{\partial v}{\partial p}\left[\mu_h l_{m_k} - \mu_k l_{m_h}\right]$$

$$(\partial m_h) = \left[\theta\frac{\partial v}{\partial \theta} - v\right]\left[\mu_k\frac{\partial v}{\partial m_y} - \mu_y\frac{\partial v}{\partial m_k}\right] + \frac{\partial v}{\partial p}\left[\mu_k l_{m_y} - \mu_y l_{m_k}\right]$$

$$(\partial m_k) = -\left[\theta\frac{\partial v}{\partial \theta} - v\right]\left[\mu_h\frac{\partial v}{\partial m_y} - \mu_y\frac{\partial v}{\partial m_h}\right] - \frac{\partial v}{\partial p}\left[\mu_h l_{m_y} - \mu_y l_{m_h}\right]$$

$$(\partial \varepsilon) = v\frac{\partial v}{\partial m_k}\left[\mu_y l_{m_h} - \mu_h l_{m_y}\right] - v\frac{\partial v}{\partial m_h}\left[\mu_y l_{m_k} - \mu_k l_{m_y}\right] - v\frac{\partial v}{\partial m_y}\left[\mu_k l_{m_h} - \mu_h l_{m_k}\right]$$

$$(\partial \eta) = v\frac{\partial v}{\partial m_k}\left[\mu_y\frac{l_{m_h}}{\theta} - \mu_h\frac{l_{m_y}}{\theta}\right] - v\frac{\partial v}{\partial m_h}\left[\mu_y\frac{l_{m_k}}{\theta} - \mu_k\frac{l_{m_y}}{\theta}\right] - v\frac{\partial v}{\partial m_y}\left[\mu_k\frac{l_{m_h}}{\theta} - \mu_h\frac{l_{m_k}}{\theta}\right]$$

$$(\partial \zeta) = -v\frac{\partial v}{\partial m_k}\left[\mu_y l_{m_h} - \mu_h l_{m_y}\right] + v\frac{\partial v}{\partial m_h}\left[\mu_y l_{m_k} - \mu_k l_{m_y}\right] + v\frac{\partial v}{\partial m_y}\left[\mu_k l_{m_h} - \mu_h l_{m_k}\right]$$

$$(dW) = 0$$

$$(dQ) = v\frac{\partial v}{\partial m_k}\left[\mu_y l_{m_h} - \mu_h l_{m_y}\right] - v\frac{\partial v}{\partial m_h}\left[\mu_y l_{m_k} - \mu_k l_{m_y}\right] - v\frac{\partial v}{\partial m_y}\left[\mu_k l_{m_h} - \mu_h l_{m_k}\right]$$

Group 118

θ, $\mathbf{m}_b$, $\boldsymbol{\varepsilon}$, $\mathfrak{n}$, ζ constant.

$$(\partial p) = p\frac{\partial v}{\partial m_k}\left[l_{m_y}\mu_h - l_{m_h}\mu_y\right] - p\frac{\partial v}{\partial m_h}\left[l_{m_y}\mu_k - l_{m_k}\mu_y\right] - p\frac{\partial v}{\partial m_y}\left[l_{m_k}\mu_h - l_{m_h}\mu_k\right]$$

$$(\partial m_y) = v\left[\mu_h l_{m_k} - \mu_k l_{m_h}\right] + p\,v\left[\frac{\partial v}{\partial m_k}l_{m_h} - \frac{\partial v}{\partial m_h}l_{m_k}\right] + p\,\theta\frac{\partial v}{\partial\theta}\left[\mu_h\frac{\partial v}{\partial m_k} - \mu_k\frac{\partial v}{\partial m_h}\right] + p\frac{\partial v}{\partial p}\left[\mu_h l_{m_k} - \mu_k l_{m_h}\right]$$

$$(\partial m_h) = \left(p\frac{\partial v}{\partial p} + v\right)\left[\mu_k l_{m_y} - \mu_y l_{m_k}\right] + p\,v\left[\frac{\partial v}{\partial m_y}l_{m_k} - \frac{\partial v}{\partial m_k}l_{m_y}\right] + p\,\theta\frac{\partial v}{\partial\theta}\left[\mu_k\frac{\partial v}{\partial m_y} - \mu_y\frac{\partial v}{\partial m_k}\right]$$

$$(\partial m_k) = -\left(p\frac{\partial v}{\partial p} + v\right)\left[\mu_h l_{m_y} - \mu_y l_{m_h}\right] - p\,v\left[\frac{\partial v}{\partial m_y}l_{m_h} - \frac{\partial v}{\partial m_h}l_{m_y}\right] - p\,\theta\frac{\partial v}{\partial\theta}\left[\mu_h\frac{\partial v}{\partial m_y} - \mu_y\frac{\partial v}{\partial m_h}\right]$$

$$(\partial v) = v\frac{\partial v}{\partial m_k}\left[l_{m_h}\mu_y - l_{m_y}\mu_h\right] - v\frac{\partial v}{\partial m_h}\left[l_{m_k}\mu_y - l_{m_y}\mu_k\right] - v\frac{\partial v}{\partial m_y}\left[l_{m_h}\mu_k - l_{m_k}\mu_h\right]$$

χ is constant.

ψ is constant.

$$(dW) = p\,v\frac{\partial v}{\partial m_k}\left[l_{m_y}\mu_h - l_{m_h}\mu_y\right] - p\,v\frac{\partial v}{\partial m_h}\left[l_{m_y}\mu_k - l_{m_k}\mu_y\right] - p\,v\frac{\partial v}{\partial m_y}\left[l_{m_k}\mu_h - l_{m_h}\mu_k\right]$$

$$(dQ) = 0$$

Group 119

θ, m_b, ε, n, χ constant.

ζ and ψ are constant. Same as *Group 118.*

Group 120

θ, m_b, ε, n, ψ constant.

Indeterminate.

Group 121

θ, m_b, ε, ζ, χ constant.

n and ψ are constant. Same as *Group 118.*

Group 122

θ, m_b, ε, ζ, ψ constant.

n and χ are constant. Same as *Group 118.*

Group 123

θ, m_b, ε, χ, ψ constant.

n and ζ are constant. Same as *Group 118.*

Group 124

θ, m_b, n, ζ, χ constant.

Indeterminate.

Group 125

θ, m_b, n, ζ, ψ constant.

ε and χ are constant. Same as *Group 118.*

Group 126

θ, m_b, n, χ, ψ constant.

ε and ζ are constant. Same as *Group 118.*

Group 127

θ, m_b, ζ, χ, ψ constant.

ε and n are constant. Same as *Group 118.*

Group 128

p, $\mathbf{m}_b$, $\mathbf{v}$, $\boldsymbol{\varepsilon}$, $\mathfrak{n}$ constant.

$$(\partial\theta) = \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[l_{m_h}\,\mu_y - \mu_h\,l_{m_y}\right] - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_k}\,\mu_y - l_{m_y}\,\mu_k\right] - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_h}\,\mu_k - l_{m_k}\,\mu_h\right]$$

$$(\partial \mathbf{m}_y) = \frac{\partial \mathbf{v}}{\partial \theta}(\mu_k\,l_{m_h} - \mu_h\,l_{m_k}) - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left[\mu_k\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \mu_h\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right]$$

$$(\partial \mathbf{m}_h) = \frac{\partial \mathbf{v}}{\partial \theta}(\mu_y\,l_{m_k} - \mu_k\,l_{m_y}) - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left[\mu_y\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\right]$$

$$(\partial \mathbf{m}_k) = -\frac{\partial \mathbf{v}}{\partial \theta}(\mu_y\,l_{m_h} - \mu_h\,l_{m_y}) + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left[\mu_y\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \mu_h\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\right]$$

$$(\partial\zeta) = \mathfrak{n}\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[\mu_h\,l_{m_y} - l_{m_h}\,\mu_y\right] - \mathfrak{n}\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_y}\,\mu_k - l_{m_k}\,\mu_y\right] - \mathfrak{n}\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_k}\,\mu_h - l_{m_h}\,\mu_k\right]$$

χ is constant.

$$(\partial\psi) = \mathfrak{n}\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[\mu_h\,l_{m_y} - l_{m_h}\,\mu_y\right] - \mathfrak{n}\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_y}\,\mu_k - l_{m_k}\,\mu_y\right] - \mathfrak{n}\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_k}\,\mu_h - l_{m_h}\,\mu_k\right]$$

$$(d\mathbf{W}) = 0$$

$$(d\mathbf{Q}) = 0$$

Group 129

p, $\mathrm{m_b}$, v, ε, ζ constant.

$$(\partial\theta) = \frac{\partial v}{\partial m_k}\left[l_{m_y}\,\mu_h - l_{m_h}\,\mu_y\right] - \frac{\partial v}{\partial m_h}\left[l_{m_y}\,\mu_k - l_{m_k}\,\mu_y\right] - \frac{\partial v}{\partial m_y}\left[l_{m_k}\,\mu_h - l_{m_h}\,\mu_k\right]$$

$$(\partial m_y) = \frac{\partial v}{\partial \theta}(\mu_h\,l_{m_k} - \mu_k\,l_{m_h}) - (m_1 + \cdots + m_n)\,c_p\left[\mu_h\,\frac{\partial v}{\partial m_k} - \mu_k\,\frac{\partial v}{\partial m_h}\right] + \mathfrak{n}\left[\frac{\partial v}{\partial m_h}(\mu_k + l_{m_k}) - \frac{\partial v}{\partial m_k}(\mu_h + l_{m_h})\right]$$

$$(\partial m_h) = \frac{\partial v}{\partial \theta}(\mu_k\,l_{m_y} - \mu_y\,l_{m_k}) - (m_1 + \cdots + m_n)\,c_p\left[\mu_k\,\frac{\partial v}{\partial m_y} - \mu_y\,\frac{\partial v}{\partial m_k}\right] + \mathfrak{n}\left[\frac{\partial v}{\partial m_k}(\mu_y + l_{m_y}) - \frac{\partial v}{\partial m_y}(\mu_k + l_{m_k})\right]$$

$$(\partial m_k) = -\frac{\partial v}{\partial \theta}(\mu_h\,l_{m_y} - \mu_y\,l_{m_h}) + (m_1 + \cdots + m_n)\,c_p\left[\mu_h\,\frac{\partial v}{\partial m_y} - \mu_y\,\frac{\partial v}{\partial m_h}\right] - \mathfrak{n}\left[\frac{\partial v}{\partial m_h}(\mu_y + l_{m_y}) - \frac{\partial v}{\partial m_y}(\mu_h + l_{m_h})\right]$$

$$(\partial \mathfrak{n}) = \mathfrak{n}\,\frac{\partial v}{\partial m_k}\left[\frac{l_{m_h}}{\theta}\,\mu_y - \frac{l_{m_y}}{\theta}\,\mu_h\right] - \mathfrak{n}\,\frac{\partial v}{\partial m_h}\left[\frac{l_{m_k}}{\theta}\,\mu_y - \frac{l_{m_y}}{\theta}\,\mu_k\right] - \mathfrak{n}\,\frac{\partial v}{\partial m_y}\left[\frac{l_{m_h}}{\theta}\,\mu_k - \frac{l_{m_k}}{\theta}\,\mu_h\right]$$

χ is constant.

ψ is constant.

$$(dW) = 0$$

$$(dQ) = \mathfrak{n}\,\frac{\partial v}{\partial m_k}\left[l_{m_h}\,\mu_y - l_{m_y}\,\mu_h\right] - \mathfrak{n}\,\frac{\partial v}{\partial m_h}\left[l_{m_k}\,\mu_y - l_{m_y}\,\mu_k\right] - \mathfrak{n}\,\frac{\partial v}{\partial m_y}\left[l_{m_h}\,\mu_k - l_{m_k}\,\mu_h\right]$$

Group 130

p, $\mathrm{m_b}$, v, ε, χ constant.

Indeterminate.

Group 131

p, $\mathbf{m}_b$, $\mathbf{v}$, ε, ψ constant.

ζ and χ are constant. Same as *Group 129.*

Group 132

p, $\mathbf{m}_b$, $\mathbf{v}$, $\mathfrak{n}$, ζ constant.

$$(\partial\theta) = \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[l_{m_y}\,\mu_h - l_{m_h}\,\mu_y\right] - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_y}\,\mu_k - l_{m_k}\,\mu_y\right] - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_k}\,\mu_h - l_{m_h}\,\mu_k\right]$$

$$(\partial \mathbf{m}_y) = \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h\, l_{m_k} - \mu_k\, l_{m_h}\right] + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] - \mathfrak{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} l_{m_h} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} l_{m_k}\right]$$

$$(\partial \mathbf{m}_h) = \frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k\, l_{m_y} - \mu_y\, l_{m_k}\right] + (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left[\mu_k \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y} - \mu_y \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] - \mathfrak{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y} l_{m_k} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} l_{m_y}\right]$$

$$(\partial \mathbf{m}_k) = -\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_h\, l_{m_y} - \mu_y\, l_{m_h}\right] - (\mathbf{m}_1 + \cdots + \mathbf{m}_n)\, c_p\left[\mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y} - \mu_y \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right] + \mathfrak{n}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y} l_{m_h} - \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} l_{m_y}\right]$$

$$(\partial\varepsilon) = \mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[l_{m_y}\,\mu_h - l_{m_h}\,\mu_y\right] - \mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_y}\,\mu_k - l_{m_k}\,\mu_y\right] - \mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_k}\,\mu_h - l_{m_h}\,\mu_k\right]$$

$$(\partial\chi) = \mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[l_{m_y}\,\mu_h - l_{m_h}\,\mu_y\right] - \mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_y}\,\mu_k - l_{m_k}\,\mu_y\right] - \mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_k}\,\mu_h - l_{m_h}\,\mu_k\right]$$

ψ is constant.

$$(d\mathbf{W}) = 0$$

$$(d\mathbf{Q}) = 0$$

Group 133

p, $\mathbf{m}_b$, $\mathbf{v}$, $\mathfrak{n}$, χ constant.

ε is constant. Same as *Group 128.*

Group 134

p, $\mathbf{m}_b$, $\mathbf{v}$, $\mathfrak{n}$, ψ constant.

ζ is constant. Same as *Group 132.*

Group 135

p, $\mathbf{m}_b$, $\mathbf{v}$, ζ, χ constant.

ε and ψ are constant. Same as *Group 129.*

Group 136

p, $\mathbf{m}_b$, $\mathbf{v}$, ζ, ψ constant.

Indeterminate.

Group 137

p, $\mathbf{m}_b$, $\mathbf{v}$, χ, ψ constant.

ε and ζ are constant. Same as *Group 129.*

Group 138

p, $\mathbf{m}_b$, ε, $\mathfrak{n}$, ζ constant.

$$(\partial\theta) = p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[l_{m_h}\mu_y - l_{m_y}\mu_h\right] - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_k}\mu_y - l_{m_y}\mu_k\right] - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_h}\mu_k - l_{m_k}\mu_h\right]$$

$$(\partial \mathbf{m}_y) = p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_k l_{m_h} - \mu_h l_{m_k}\right] - p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left[\mu_k\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \mu_h\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right] + \mathfrak{n}\left[l_{m_k}\left(\mu_h - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right) - l_{m_h}\left(\mu_k - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)\right]$$

$$(\partial \mathbf{m}_h) = p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_y l_{m_k} - \mu_k l_{m_y}\right] - p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,c_p\left[\mu_y\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\right] + \mathfrak{n}\left[l_{m_y}\left(\mu_k - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right) - l_{m_k}\left(\mu_y - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\right)\right]$$

Group 138 (Con.)

$$(\partial \mathbf{m}_k) = -p\frac{\partial \mathbf{v}}{\partial \theta}\left[\mu_y l_{m_h} - \mu_h l_{m_y}\right] + p(\mathbf{m}_1 + \cdots + \mathbf{m}_n)c_p\left[\mu_y \frac{\partial \mathbf{v}}{\partial \mathbf{m}_h} - \mu_h \frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\right] - \mathfrak{n}\left[l_{m_y}\left(\mu_h - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right) - l_{m_h}\left(\mu_y - p\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\right)\right]$$

$$(\partial \mathbf{v}) = \mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[l_{m_h}\mu_y - l_{m_y}\mu_h\right] - \mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_k}\mu_y - l_{m_y}\mu_k\right] - \mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_h}\mu_k - l_{m_k}\mu_h\right]$$

$$(\partial \chi) = p\,\mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[l_{m_h}\mu_y - l_{m_y}\mu_h\right] - p\,\mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_k}\mu_y - l_{m_y}\mu_k\right] - p\,\mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_h}\mu_k - l_{m_k}\mu_h\right]$$

$$(\partial \psi) = p\,\mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[l_{m_y}\mu_h - l_{m_h}\mu_y\right] - p\,\mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_y}\mu_k - l_{m_k}\mu_y\right] - p\,\mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_k}\mu_h - l_{m_h}\mu_k\right]$$

$$(d\mathbf{W}) = p\,\mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[l_{m_y}\mu_h - l_{m_h}\mu_y\right] - p\,\mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_y}\mu_k - l_{m_k}\mu_y\right] - p\,\mathfrak{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_k}\mu_h - l_{m_h}\mu_k\right]$$

$$(d\mathbf{Q}) = 0$$

Group 139

p, $\mathbf{m}_b$, $\boldsymbol{\varepsilon}$, $\mathfrak{n}$, χ constant.

$\mathbf{v}$ is constant. Same as *Group 128.*

Group 140

p, $\mathbf{m}_b$, $\boldsymbol{\varepsilon}$, $\mathfrak{n}$, ψ constant.

θ is constant. Same as *Group 98.*

Group 141

p, $\mathbf{m}_b$, $\boldsymbol{\varepsilon}$, ζ, χ constant.

$\mathbf{v}$ and ψ are constant. Same as *Group 129.*

Group 142

p, $\mathbf{m}_b$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\zeta}$, $\boldsymbol{\psi}$ constant.

$\mathbf{v}$ and $\boldsymbol{\chi}$ are constant. Same as *Group 129.*

Group 143

p, $\mathbf{m}_b$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\chi}$, $\boldsymbol{\psi}$ constant.

$\mathbf{v}$ and $\boldsymbol{\zeta}$ are constant. Same as *Group 129.*

Group 144

p, $\mathbf{m}_b$, $\mathbf{n}$, $\boldsymbol{\zeta}$, $\boldsymbol{\chi}$ constant.

θ is constant. Same as *Group 102.*

Group 145

p, $\mathbf{m}_b$, $\mathbf{n}$, $\boldsymbol{\zeta}$, $\boldsymbol{\psi}$ constant.

$\mathbf{v}$ is constant. Same as *Group 132.*

Group 146

p, $\mathbf{m}_b$, $\mathbf{n}$, $\boldsymbol{\chi}$, $\boldsymbol{\psi}$ constant.

$$(\partial\theta) = p\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[\frac{l_{m_h}}{\theta}\,\mu_y - \frac{l_{m_y}}{\theta}\,\mu_h\right] - p\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[\frac{l_{m_k}}{\theta}\,\mu_y - \frac{l_{m_y}}{\theta}\,\mu_k\right] - p\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[\frac{l_{m_h}}{\theta}\,\mu_k - \frac{l_{m_k}}{\theta}\,\mu_h\right]$$

$$(\partial \mathbf{m}_y) = \left(\mathbf{n} + p\,\frac{\partial \mathbf{v}}{\partial \theta}\right)\left[\mu_k\,\frac{l_{m_h}}{\theta} - \mu_h\,\frac{l_{m_k}}{\theta}\right] + p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,\frac{c_p}{\theta}\left[\mu_h\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k} - \mu_k\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right]$$

$$(\partial \mathbf{m}_h) = \left(\mathbf{n} + p\,\frac{\partial \mathbf{v}}{\partial \theta}\right)\left[\mu_y\,\frac{l_{m_k}}{\theta} - \mu_k\,\frac{l_{m_y}}{\theta}\right] + p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,\frac{c_p}{\theta}\left[\mu_k\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y} - \mu_y\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right]$$

$$(\partial \mathbf{m}_k) = -\left(\mathbf{n} + p\,\frac{\partial \mathbf{v}}{\partial \theta}\right)\left[\mu_y\,\frac{l_{m_h}}{\theta} - \mu_h\,\frac{l_{m_y}}{\theta}\right] - p\,(\mathbf{m}_1 + \cdots + \mathbf{m}_n)\,\frac{c_p}{\theta}\left[\mu_h\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y} - \mu_y\,\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right]$$

Group 146 (Con.)

$$(\partial \mathbf{v}) = \mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[\frac{l_{m_y}}{\theta}\mu_h - \frac{l_{m_h}}{\theta}\mu_y\right] - \mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[\frac{l_{m_y}}{\theta}\mu_k - \frac{l_{m_k}}{\theta}\mu_y\right] - \mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[\frac{l_{m_k}}{\theta}\mu_h - \frac{l_{m_h}}{\theta}\mu_k\right]$$

$$(\partial \boldsymbol{\varepsilon}) = p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[\frac{l_{m_h}}{\theta}\mu_y - \frac{l_{m_y}}{\theta}\mu_h\right] - p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[\frac{l_{m_k}}{\theta}\mu_y - \frac{l_{m_y}}{\theta}\mu_k\right] - p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[\frac{l_{m_h}}{\theta}\mu_k - \frac{l_{m_k}}{\theta}\mu_h\right]$$

$$(\partial \zeta) = p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[\frac{l_{m_y}}{\theta}\mu_h - \frac{l_{m_h}}{\theta}\mu_y\right] - p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[\frac{l_{m_y}}{\theta}\mu_k - \frac{l_{m_k}}{\theta}\mu_y\right] - p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[\frac{l_{m_k}}{\theta}\mu_h - \frac{l_{m_h}}{\theta}\mu_k\right]$$

$$(d\mathbf{W}) = p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[\frac{l_{m_h}}{\theta}\mu_y - \frac{l_{m_y}}{\theta}\mu_h\right] - p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[\frac{l_{m_k}}{\theta}\mu_y - \frac{l_{m_y}}{\theta}\mu_k\right] - p\,\mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[\frac{l_{m_h}}{\theta}\mu_k - \frac{l_{m_k}}{\theta}\mu_h\right]$$

$$(d\mathbf{Q}) = 0$$

Group 147

p, $\mathbf{m}_b$, ζ, χ, ψ constant.

$\mathbf{v}$ and $\boldsymbol{\varepsilon}$ are constant. Same as *Group 129.*

Group 148

$\mathbf{m}_b$, $\mathbf{v}$, $\boldsymbol{\varepsilon}$, $\mathbf{n}$, ζ constant.

$$(\partial \theta) = \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[l_{m_y}\mu_h - l_{m_h}\mu_y\right] - \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[l_{m_y}\mu_k - l_{m_k}\mu_y\right] - \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[l_{m_k}\mu_h - l_{m_h}\mu_k\right]$$

$$(\partial p) = \mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\left[\mu_h l_{m_y} - \mu_y l_{m_h}\right] - \mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\left[\mu_k l_{m_y} - \mu_y l_{m_k}\right] - \mathbf{n}\frac{\partial \mathbf{v}}{\partial \mathbf{m}_y}\left[\mu_h l_{m_k} - \mu_k l_{m_h}\right]$$

Group 148 (Con.)

$$(\partial m_y) = \left[v\frac{\partial v}{\partial \theta} + n\frac{\partial v}{\partial p}\right]\left[\mu_h\, l_{m_k} - \mu_k\, l_{m_h}\right] - \left[\theta\, n\frac{\partial v}{\partial \theta} - v\,(m_1 + \cdots + m_n)\, c_p\right]\left[\mu_k\frac{\partial v}{\partial m_h} - \mu_h\frac{\partial v}{\partial m_k}\right]$$

$$(\partial m_h) = \left[v\frac{\partial v}{\partial \theta} + n\frac{\partial v}{\partial p}\right]\left[\mu_k\, l_{m_y} - \mu_y\, l_{m_k}\right] - \left[\theta\, n\frac{\partial v}{\partial \theta} - v\,(m_1 + \cdots + m_n)\, c_p\right]\left[\mu_y\frac{\partial v}{\partial m_k} - \mu_k\frac{\partial v}{\partial m_y}\right]$$

$$(\partial m_k) = -\left[v\frac{\partial v}{\partial \theta} + n\frac{\partial v}{\partial p}\right]\left[\mu_h\, l_{m_y} - \mu_y\, l_{m_h}\right] + \left[\theta\, n\frac{\partial v}{\partial \theta} - v\,(m_1 + \cdots + m_n)\, c_p\right]\left[\mu_y\frac{\partial v}{\partial m_h} - \mu_h\frac{\partial v}{\partial m_y}\right]$$

$$(\partial \chi) = v\, n\frac{\partial v}{\partial m_k}\left[\mu_h\, l_{m_y} - \mu_y\, l_{m_h}\right] - v\, n\frac{\partial v}{\partial m_h}\left[\mu_k\, l_{m_y} - \mu_y\, l_{m_k}\right] - v\, n\frac{\partial v}{\partial m_y}\left[\mu_h\, l_{m_k} - \mu_k\, l_{m_h}\right]$$

$$(\partial \psi) = v\, n\frac{\partial v}{\partial m_k}\left[\mu_y\, l_{m_h} - \mu_h\, l_{m_y}\right] - v\, n\frac{\partial v}{\partial m_h}\left[l_{m_k}\, \mu_y - l_{m_y}\, \mu_k\right] - v\, n\frac{\partial v}{\partial m_y}\left[\mu_k\, l_{m_h} - \mu_h\, l_{m_k}\right]$$

$$(dW) = 0$$

$$(dQ) = 0$$

Group 149

m_b, v, ε, n, χ constant.

p is constant. Same as *Group 128.*

Group 150

m_b, v, ε, n, ψ constant.

θ is constant. Same as *Group 108.*

Group 151

m_b, v, ε, ζ, χ constant.

p and ψ are constant. Same as *Group 129.*

Group 152

m_b, v, ε, ζ, ψ constant.

p and χ are constant. Same as *Group 129.*

Group 153

m_b, v, ε, χ, ψ constant.

p and ζ are constant. Same as *Group 129*

Group 154

m_b, v, n, ζ, χ constant.

θ is constant. Same as *Group 112.*

Group 155

m_b, v, n, ζ, ψ constant.

p is constant. Same as *Group 132.*

Group 156

m_b, v, n, χ, ψ constant.

$$(\partial\theta) = v\frac{\partial v}{\partial m_k}\left[\mu_h\frac{l_{m_y}}{\theta} - \mu_y\frac{l_{m_h}}{\theta}\right] - v\frac{\partial v}{\partial m_h}\left[\mu_k\frac{l_{m_y}}{\theta} - \mu_y\frac{l_{m_k}}{\theta}\right] - v\frac{\partial v}{\partial m_y}\left[\mu_h\frac{l_{m_k}}{\theta} - \mu_k\frac{l_{m_h}}{\theta}\right]$$

$$(\partial p) = n\frac{\partial v}{\partial m_k}\left[\mu_y\frac{l_{m_h}}{\theta} - \mu_h\frac{l_{m_y}}{\theta}\right] + n\frac{\partial v}{\partial m_h}\left[\mu_k\frac{l_{m_y}}{\theta} - \mu_y\frac{l_{m_k}}{\theta}\right] + n\frac{\partial v}{\partial m_y}\left[\mu_h\frac{l_{m_k}}{\theta} - \mu_k\frac{l_{m_h}}{\theta}\right]$$

Group 156 (Con.)

$$(\partial \mathrm{m_y}) = \left[\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{n}\frac{\partial \mathrm{v}}{\partial p}\right]\left[\mu_{\mathrm{h}}\frac{l_{m_\mathrm{k}}}{\theta} - \mu_{\mathrm{k}}\frac{l_{m_\mathrm{h}}}{\theta}\right] - \left[\mathrm{v}\,(\mathrm{m_1} + \cdots + \mathrm{m_n})\frac{c_p}{\theta} + \mathrm{n}\frac{\partial \mathrm{v}}{\partial \theta}\right]\left[\mu_{\mathrm{h}}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}} - \mu_{\mathrm{k}}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right] + \mathrm{n\,v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\frac{l_{m_\mathrm{k}}}{\theta} - \frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\frac{l_{m_\mathrm{h}}}{\theta}\right]$$

$$(\partial \mathrm{m_h}) = \left[\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{n}\frac{\partial \mathrm{v}}{\partial p}\right]\left[\mu_{\mathrm{k}}\frac{l_{m_\mathrm{y}}}{\theta} - \mu_{\mathrm{y}}\frac{l_{m_\mathrm{k}}}{\theta}\right] - \left[\mathrm{v}\,(\mathrm{m_1} + \cdots + \mathrm{m_n})\frac{c_p}{\theta} + \mathrm{n}\frac{\partial \mathrm{v}}{\partial \theta}\right]\left[\mu_{\mathrm{k}}\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}} - \mu_{\mathrm{y}}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right] + \mathrm{n\,v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\frac{l_{m_\mathrm{y}}}{\theta} - \frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}\frac{l_{m_\mathrm{k}}}{\theta}\right]$$

$$(\partial \mathrm{m_k}) = -\left[\mathrm{v}\frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{n}\frac{\partial \mathrm{v}}{\partial p}\right]\left[\mu_{\mathrm{h}}\frac{l_{m_\mathrm{y}}}{\theta} - \mu_{\mathrm{y}}\frac{l_{m_\mathrm{h}}}{\theta}\right] + \left[\mathrm{v}\,(\mathrm{m_1} + \cdots + \mathrm{m_n})\frac{c_p}{\theta} + \mathrm{n}\frac{\partial \mathrm{v}}{\partial \theta}\right]\left[\mu_{\mathrm{h}}\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}} - \mu_{\mathrm{y}}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right] - \mathrm{n\,v}\left[\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\frac{l_{m_\mathrm{y}}}{\theta} - \frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}\frac{l_{m_\mathrm{h}}}{\theta}\right]$$

$$(\partial \varepsilon) = \mathrm{v\,n}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[\mu_{\mathrm{h}}\frac{l_{m_\mathrm{y}}}{\theta} - \mu_{\mathrm{y}}\frac{l_{m_\mathrm{h}}}{\theta}\right] - \mathrm{v\,n}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\left[\mu_{\mathrm{k}}\frac{l_{m_\mathrm{y}}}{\theta} - \mu_{\mathrm{y}}\frac{l_{m_\mathrm{k}}}{\theta}\right] - \mathrm{v\,n}\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}\left[\mu_{\mathrm{h}}\frac{l_{m_\mathrm{k}}}{\theta} - \mu_{\mathrm{k}}\frac{l_{m_\mathrm{h}}}{\theta}\right]$$

$$(\partial \zeta) = \mathrm{v\,n}\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\left[\mu_{\mathrm{y}}\frac{l_{m_\mathrm{h}}}{\theta} - \mu_{\mathrm{h}}\frac{l_{m_\mathrm{y}}}{\theta}\right] - \mathrm{v\,n}\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\left[\mu_{\mathrm{y}}\frac{l_{m_\mathrm{k}}}{\theta} - \mu_{\mathrm{k}}\frac{l_{m_\mathrm{y}}}{\theta}\right] - \mathrm{v\,n}\frac{\partial \mathrm{v}}{\partial \mathrm{m_y}}\left[\mu_{\mathrm{k}}\frac{l_{m_\mathrm{h}}}{\theta} - \mu_{\mathrm{h}}\frac{l_{m_\mathrm{k}}}{\theta}\right]$$

$$(\mathrm{dW}) = 0$$

$$(\mathrm{dQ}) = 0$$

Group 157

m_b, v, ζ, χ, ψ constant.

p and ε are constant. Same as *Group 129.*

Group 158

m_b, ε, η, ζ, χ constant.

θ and ψ are constant. Same as *Group 118.*

Group 159

m_b, ε, η, ζ, ψ constant.

θ and χ are constant. Same as *Group 118.*

Group 160

m_b, ε, η, χ, ψ constant.

θ and ζ are constant. Same as *Group 118.*

Group 161

m_b, ε, ζ, χ, ψ constant.

Indeterminate.

Group 162

m_b, η, ζ, χ, ψ constant.

θ and ε are constant. Same as *Group 118.*

TABLE II

Second Derivatives†

*Group 1**

$\theta, \mathbf{m}_1, \cdots, \mathbf{m}_n$ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p, \mathbf{m}_a}\right]_{\theta, \mathbf{m}_a}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta, \mathbf{m}_a}\right]_{\theta, \mathbf{m}_a}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta, p, \mathbf{m}_i}\right]_{\theta, \mathbf{m}_a}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\, \mathbf{m}_a) c_p\right]_{\theta, \mathbf{m}_a} = -\theta\left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p, \mathbf{m}_a}$$

$$\left[\frac{\partial l_{m_k}}{\partial p}\right]_{\theta, \mathbf{m}_a} = -\theta\left[\frac{\partial}{\partial \mathbf{m}_k}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p, \mathbf{m}_a}\right]_{\theta, p, \mathbf{m}_i}$$

$$\left[\frac{\partial \mu_k}{\partial p}\right]_{\theta, \mathbf{m}_a} = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta, p, \mathbf{m}_i}$$

† Subscripts: $\mathbf{e} = \boldsymbol{\varepsilon}$; $\mathbf{n} = \mathfrak{n}$; $\mathbf{x} = \boldsymbol{\chi}$; $\mathbf{y} = \boldsymbol{\psi}$; $\mathbf{z} = \boldsymbol{\zeta}$, throughout Table II.

* $\mathbf{m}_a$ denotes all the component masses; $\mathbf{m}_i$ all except $\mathbf{m}_k$.

*Group 2**

$p, \mathbf{m}_1, \cdots, \mathbf{m}_n$ constant.

$$\left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\,\mathbf{m}_a}$$

$$\left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{p,\,\mathbf{m}_a}$$

$$\left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{p,\,\mathbf{m}_a}$$

$$\left[\frac{\partial}{\partial \theta}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{p,\,\mathbf{m}_a} = (\Sigma\,\mathbf{m}_a)\left(\frac{\partial\, c_p}{\partial \theta}\right)_{p,\,\mathbf{m}_a}$$

$$\left(\frac{\partial\, l_{m_k}}{\partial \theta}\right)_{p,\,\mathbf{m}_a} = \frac{l_{m_k}}{\theta} + \left(\frac{\partial\,(\Sigma\,\mathbf{m}_a)\,c_p}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}$$

$$\left(\frac{\partial \mu_k}{\partial \theta}\right)_{p,\,\mathbf{m}_a} = -\,\frac{l_{m_k}}{\theta}$$

*Group 3**

$\mathbf{m}_1, \cdots, \mathbf{m}_n, \mathbf{v}$ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\mathbf{m}_a,\,\mathbf{v}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,\mathbf{m}_a} - \left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\,\mathbf{m}_a} \frac{\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}}{\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{\mathbf{m}_a,\,\mathbf{v}} = \left(\frac{\partial^2 \mathbf{v}}{\partial p^2}\right)_{\theta,\,\mathbf{m}_a} - \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{p,\,\mathbf{m}_a} \frac{\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}}{\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\mathbf{m}_a,\,\mathbf{v}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\theta,\,\mathbf{m}_a} - \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{p,\,\mathbf{m}_a} \frac{\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}}{\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{\mathbf{m}_a,\,\mathbf{v}} = -\,\theta\left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\,\mathbf{m}_a} - (\Sigma\,\mathbf{m}_a)\left(\frac{\partial\,c_p}{\partial \theta}\right)_{p,\,\mathbf{m}_a} \frac{\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}}{\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}$$

* $\mathbf{m}_a$ denotes all the component masses; $\mathbf{m}_i$ all except $\mathbf{m}_k$.

*Group 3** (Con.)

$$\left(\frac{\partial\, l_{mk}}{\partial p}\right)_{m_a,\, v} = -\theta\left[\frac{\partial}{\partial m_k}\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right]_{\theta,\, p,\, m_i} - \left\{\frac{l_{mk}}{\theta} + \left[\frac{\partial\,(\Sigma\, m_a)\, c_p}{\partial m_k}\right]_{\theta,\, p,\, m_i}\right\}\frac{\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_a}}{\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}}$$

$$\left(\frac{\partial \mu_k}{\partial p}\right)_{m_a,\, v} = \left(\frac{\partial v}{\partial m_k}\right)_{\theta,\, p,\, m_i} + \frac{l_{mk}}{\theta}\,\frac{\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_a}}{\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}}$$

*Group 4**

$m_1, \cdots, m_n, \varepsilon$ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right]_{m_a,\, e} = \left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right]_{\theta,\, m_a} + \left(\frac{\partial^2 v}{\partial \theta^2}\right)_{p,\, m_a}\frac{\theta\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a} + p\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_a}}{(\Sigma\, m_a)\, c_p - p\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_a}\right]_{m_a,\, e} = \left(\frac{\partial^2 v}{\partial p^2}\right)_{\theta,\, m_a} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_a}\right]_{p,\, m_a}\frac{\theta\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a} + p\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_a}}{(\Sigma\, m_a)\, c_p - p\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\mathrm{m_a},\,\mathrm{e}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{p,\,\mathrm{m_a}} \frac{\theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}} + p\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}}{(\Sigma\,\mathrm{m_a})\,c_p - p\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\mathrm{m_a},\,\mathrm{e}} = -\,\theta\left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}} + (\Sigma\,\mathrm{m_a})\left(\frac{\partial\,c_p}{\partial \theta}\right)_{p,\,\mathrm{m_a}} \frac{\theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}} + p\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}}{(\Sigma\,\mathrm{m_a})\,c_p - p\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}}$$

$$\left(\frac{\partial\,l_{m\mathrm{k}}}{\partial p}\right)_{\mathrm{m_a},\,\mathrm{e}} = -\,\theta\left[\frac{\partial}{\partial \mathrm{m_k}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_i}} + \left[\frac{l_{m\mathrm{k}}}{\theta} + \left(\frac{\partial\,(\Sigma\,\mathrm{m_a})\,c_p}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right] \frac{\theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}} + p\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}}{(\Sigma\,\mathrm{m_a})\,c_p - p\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}}$$

$$\left(\frac{\partial \mu_{\mathrm{k}}}{\partial p}\right)_{\mathrm{m_a},\,\mathrm{e}} = \left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}} - \frac{l_{m\mathrm{k}}}{\theta}\,\frac{\left[\theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}} + p\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]}{\left[(\Sigma\,\mathrm{m_a})\,c_p - p\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]}$$

* $\mathrm{m_a}$ denotes all the component masses; $\mathrm{m_i}$ all except $\mathrm{m_k}$.

*Group 5**

$\mathbf{m}_1, \cdots, \mathbf{m}_n, \mathbf{n}$ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\mathbf{m}_a,\,\mathbf{n}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,\mathbf{m}_a} + \left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\,\mathbf{m}_a} \frac{\theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}{(\Sigma\, \mathbf{m}_a)\, c_p}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{\mathbf{m}_a,\,\mathbf{n}} = \left(\frac{\partial^2 \mathbf{v}}{\partial p^2}\right)_{\theta,\,\mathbf{m}_a} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{p,\,\mathbf{m}_a} \frac{\theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}{(\Sigma\, \mathbf{m}_a)\, c_p}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\mathbf{m}_a,\,\mathbf{n}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\theta,\,\mathbf{m}_a} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{p,\,\mathbf{m}_a} \frac{\theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}{(\Sigma\, \mathbf{m}_a)\, c_p}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\, \mathbf{m}_a)\, c_p\right]_{\mathbf{m}_a,\,\mathbf{n}} = -\,\theta\left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\,\mathbf{m}_a} + (\Sigma\, \mathbf{m}_a)\left(\frac{\partial\, c_p}{\partial \theta}\right)_{p,\,\mathbf{m}_a} \frac{\theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}{(\Sigma\, \mathbf{m}_a)\, c_p}$$

$$\left(\frac{\partial\, l_{m_k}}{\partial p}\right)_{\mathbf{m}_a,\,\mathbf{n}} = -\,\theta\left[\frac{\partial}{\partial \mathbf{m}_k}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_i} + \left[\frac{l_{m_k}}{\theta} + \left(\frac{\partial\,(\Sigma\, \mathbf{m}_a)\, c_p}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right] \frac{\theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}{(\Sigma\, \mathbf{m}_a)\, c_p}$$

$$\left(\frac{\partial \mu_k}{\partial p}\right)_{\mathbf{m}_a,\,\mathbf{n}} = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i} - \frac{l_{m_k}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}{(\Sigma\, \mathbf{m}_a)\, c_p}$$

*Group 6**

$\mathbf{m}_1, \cdots, \mathbf{m}_n, \zeta$ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\, \mathbf{m}_a}\right]_{\mathbf{m}_a,\, z} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\, \mathbf{m}_a}\right]_{\theta,\, \mathbf{m}_a} + \left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\, \mathbf{m}_a} \frac{\mathbf{v}}{\mathfrak{n}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\, \mathbf{m}_a}\right]_{\mathbf{m}_a,\, z} = \left(\frac{\partial^2 \mathbf{v}}{\partial p^2}\right)_{\theta,\, \mathbf{m}_a} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\, \mathbf{m}_a}\right]_{p,\, \mathbf{m}_a} \frac{\mathbf{v}}{\mathfrak{n}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\, p,\, \mathbf{m}_i}\right]_{\mathbf{m}_a,\, z} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\, p,\, \mathbf{m}_i}\right]_{\theta,\, \mathbf{m}_a} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\, p,\, \mathbf{m}_i}\right]_{p,\, \mathbf{m}_a} \frac{\mathbf{v}}{\mathfrak{n}}$$

$$\left[\frac{\partial}{\partial p}\, (\Sigma\, \mathbf{m}_a)\, c_p\right]_{\mathbf{m}_a,\, z} = -\,\theta \left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\, \mathbf{m}_a} + (\Sigma\, \mathbf{m}_a) \left(\frac{\partial\, c_p}{\partial \theta}\right)_{p,\, \mathbf{m}_a} \frac{\mathbf{v}}{\mathfrak{n}}$$

$$\left(\frac{\partial\, l_{m_k}}{\partial p}\right)_{\mathbf{m}_a,\, z} = -\,\theta \left[\frac{\partial}{\partial \mathbf{m}_k}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\, \mathbf{m}_a}\right]_{\theta,\, p,\, \mathbf{m}_i} + \left[\frac{l_{m_k}}{\theta} + \left(\frac{\partial\, (\Sigma\, \mathbf{m}_a)\, c_p}{\partial \mathbf{m}_k}\right)_{\theta,\, p,\, \mathbf{m}_i}\right] \frac{\mathbf{v}}{\mathfrak{n}}$$

$$\left(\frac{\partial \mu_k}{\partial p}\right)_{\mathbf{m}_a,\, z} = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\, p,\, \mathbf{m}_i} - \frac{l_{m_k}\, \mathbf{v}}{\theta\, \mathfrak{n}}$$

* $\mathbf{m}_a$ denotes all the component masses; $\mathbf{m}_i$ all except $\mathbf{m}_k$.

*Group 7**

$\mathbf{m}_1, \cdots, \mathbf{m}_n, \chi$ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\mathbf{m}_a,\,\chi} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,\mathbf{m}_a} - \left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\,\mathbf{m}_a} \frac{\mathbf{v} - \theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}{(\Sigma\,\mathbf{m}_a)\,c_p}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{\mathbf{m}_a,\,\chi} = \left(\frac{\partial^2 \mathbf{v}}{\partial p^2}\right)_{\theta,\,\mathbf{m}_a} - \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{p,\,\mathbf{m}_a} \frac{\mathbf{v} - \theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}{(\Sigma\,\mathbf{m}_a)\,c_p}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\mathbf{m}_a,\,\chi} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\theta,\,\mathbf{m}_a} - \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{p,\,\mathbf{m}_a} \frac{\mathbf{v} - \theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}{(\Sigma\,\mathbf{m}_a)\,c_p}$$

$$\left[\frac{\partial}{\partial p}\,(\Sigma\,\mathbf{m}_a)\,c_p\right]_{\mathbf{m}_a,\,\chi} = -\,\theta\left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\,\mathbf{m}_a} - (\Sigma\,\mathbf{m}_a)\left(\frac{\partial\,c_p}{\partial \theta}\right)_{p,\,\mathbf{m}_a} \frac{\mathbf{v} - \theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}{(\Sigma\,\mathbf{m}_a)\,c_p}$$

$$\left(\frac{\partial\,l_{m_k}}{\partial p}\right)_{\mathbf{m}_a,\,\chi} = -\,\theta\left[\frac{\partial}{\partial \mathbf{m}_k}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_i} - \left[\frac{l_{m_k}}{\theta} + \left(\frac{\partial\,(\Sigma\,\mathbf{m}_a)\,c_p}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]\left[\frac{\mathbf{v} - \theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}{(\Sigma\,\mathbf{m}_a)\,c_p}\right]$$

$$\left(\frac{\partial \mu_k}{\partial p}\right)_{\mathbf{m}_a,\,\chi} = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i} + \frac{l_{m_k}}{\theta}\left[\frac{\mathbf{v} - \theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}{(\Sigma\,\mathbf{m}_a)\,c_p}\right]$$

*Group 8**

$\mathbf{m}_1, \cdots, \mathbf{m}_n, \psi$ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p, \mathbf{m}_a}\right]_{\mathbf{m}_a, \mathbf{y}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p, \mathbf{m}_a}\right]_{\theta, \mathbf{m}_a} - \left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p, \mathbf{m}_a} \frac{p\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta, \mathbf{m}_a}}{p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p, \mathbf{m}_a} + \mathbf{n}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta, \mathbf{m}_a}\right]_{\mathbf{m}_a, \mathbf{y}} = \left(\frac{\partial^2 \mathbf{v}}{\partial p^2}\right)_{\theta, \mathbf{m}_a} - \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta, \mathbf{m}_a}\right]_{p, \mathbf{m}_a} \frac{p\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta, \mathbf{m}_a}}{p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p, \mathbf{m}_a} + \mathbf{n}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta, p, \mathbf{m}_i}\right]_{\mathbf{m}_a, \mathbf{y}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta, p, \mathbf{m}_i}\right]_{\theta, \mathbf{m}_a} - \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta, p, \mathbf{m}_i}\right]_{p, \mathbf{m}_a} \frac{p\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta, \mathbf{m}_a}}{p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p, \mathbf{m}_a} + \mathbf{n}}$$

$$\left(\frac{\partial c_p}{\partial p}\right)_{\mathbf{m}_a, \mathbf{y}} = -\theta\left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p, \mathbf{m}_a} - (\Sigma \mathbf{m}_a)\left(\frac{\partial c_p}{\partial \theta}\right)_{p, \mathbf{m}_a} \frac{p\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta, \mathbf{m}_a}}{p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p, \mathbf{m}_a} + \mathbf{n}}$$

$$\left(\frac{\partial l_{m_k}}{\partial p}\right)_{\mathbf{m}_a, \mathbf{y}} = -\theta\left[\frac{\partial}{\partial \mathbf{m}_k}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p, \mathbf{m}_a}\right]_{\theta, p, \mathbf{m}_i} - \left[\frac{l_{m_k}}{\theta} + \left(\frac{\partial (\Sigma \mathbf{m}_a) c_p}{\partial \mathbf{m}_k}\right)_{\theta, p, \mathbf{m}_i}\right] \frac{p\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta, \mathbf{m}_a}}{p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p, \mathbf{m}_a} + \mathbf{n}}$$

* $\mathbf{m}_a$ denotes all the component masses; $\mathbf{m}_i$ all except $\mathbf{m}_k$.

*Group 8** (Con.)

$$\left(\frac{\partial \mu_k}{\partial p}\right)_{m_a,\,y} = \left(\frac{\partial v}{\partial m_k}\right)_{\theta,\,p,\,m_i} + \frac{l_{m_k}\left[p\left(\frac{\partial v}{\partial p}\right)_{\theta,\,m_a}\right]}{\theta\left[p\left(\frac{\partial v}{\partial \theta}\right)_{p,\,m_a} + \mathbf{n}\right]}$$

Group 9†

θ, p, m_j constant.

$$\left[\frac{\partial}{\partial m_h}\left(\frac{\partial v}{\partial \theta}\right)_{p,\,m_a}\right]_{\theta,\,p,\,m_j}$$

$$\left[\frac{\partial}{\partial m_h}\left(\frac{\partial v}{\partial p}\right)_{\theta,\,m_a}\right]_{\theta,\,p,\,m_j}$$

$$\left[\frac{\partial}{\partial m_h}\left(\frac{\partial v}{\partial m_k}\right)_{\theta,\,p,\,m_i}\right]_{\theta,\,p,\,m_j}$$

$$\left[\frac{\partial}{\partial m_h}(\Sigma\, m_s)\, c_p\right]_{\theta,\,p,\,m_j}$$

$$\left(\frac{\partial\, l_{m_k}}{\partial m_h}\right)_{\theta,\,p,\,m_j}$$

$$\left(\frac{\partial \mu_k}{\partial m_h}\right)_{\theta,\,p,\,m_j}$$

*Group 10**

θ, $\mathrm{m_j}$, v constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,\mathrm{m_j},\,\mathrm{v}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,\mathrm{m_a}} - \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{\left(\dfrac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}}{\left(\dfrac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\theta,\,\mathrm{m_j},\,\mathrm{v}} = \left(\frac{\partial^2 \mathrm{v}}{\partial p^2}\right)_{\theta,\,\mathrm{m_a}} - \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{\left(\dfrac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}}{\left(\dfrac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,\mathrm{m_j},\,\mathrm{v}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,\mathrm{m_a}} - \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{\left(\dfrac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}}{\left(\dfrac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,\mathrm{m_j},\,\mathrm{v}} = -\,\theta\left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}} - \left(\frac{\partial}{\partial \mathrm{m_h}}(\Sigma\,\mathrm{m_a})\,c_p\right)_{\theta,\,p,\,\mathrm{m_j}} \frac{\left(\dfrac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}}{\left(\dfrac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

* $\mathrm{m_a}$ denotes all the component masses; $\mathrm{m_j}$ all except $\mathrm{m_h}$; $\mathrm{m_i}$ all except $\mathrm{m_k}$.

*Group 10** (Con.)

$$\left[\frac{\partial\, l_{m_k}}{\partial p}\right]_{\theta,\, m_j,\, v} = -\,\theta\left[\frac{\partial}{\partial m_k}\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right]_{\theta,\, p,\, m_i} - \left(\frac{\partial\, l_{m_k}}{\partial m_h}\right)_{\theta,\, p,\, m_j} \frac{\left(\dfrac{\partial v}{\partial p}\right)_{\theta,\, m_a}}{\left(\dfrac{\partial v}{\partial m_h}\right)_{\theta,\, p,\, m_j}}$$

$$\left(\frac{\partial \mu_k}{\partial p}\right)_{\theta,\, m_j,\, v} = \left(\frac{\partial v}{\partial m_k}\right)_{\theta,\, p,\, m_i} - \left(\frac{\partial \mu_k}{\partial m_h}\right)_{\theta,\, p,\, m_j} \frac{\left(\dfrac{\partial v}{\partial p}\right)_{\theta,\, m_a}}{\left(\dfrac{\partial v}{\partial m_h}\right)_{\theta,\, p,\, m_j}}$$

*Group 11**

θ, m_j, ε constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right]_{\theta,\, m_j,\, e} = \left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right]_{\theta,\, m_a} + \left[\frac{\partial}{\partial m_h}\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right]_{\theta,\, p,\, m_j} \frac{\theta\left(\dfrac{\partial v}{\partial \theta}\right)_{p,\, m_a} + p\left(\dfrac{\partial v}{\partial p}\right)_{\theta,\, m_a}}{\mu_h + l_{m_h} - p\left(\dfrac{\partial v}{\partial m_h}\right)_{\theta,\, p,\, m_j}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_a}\right]_{\theta,\, m_j,\, e} = \left(\frac{\partial^2 v}{\partial p^2}\right)_{\theta,\, m_a} + \left[\frac{\partial}{\partial m_h}\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_a}\right]_{\theta,\, p,\, m_j} \frac{\theta\left(\dfrac{\partial v}{\partial \theta}\right)_{p,\, m_a} + p\left(\dfrac{\partial v}{\partial p}\right)_{\theta,\, m_a}}{\mu_h + l_{m_h} - p\left(\dfrac{\partial v}{\partial m_h}\right)_{\theta,\, p,\, m_j}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\, p,\, \mathrm{m_i}}\right]_{\theta,\, \mathrm{m_j},\, \mathrm{e}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\, p,\, \mathrm{m_i}}\right]_{\theta,\, \mathrm{m_a}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\, p,\, \mathrm{m_i}}\right]_{\theta,\, p,\, \mathrm{m_j}} \frac{\theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\, \mathrm{m_a}} + p\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\, \mathrm{m_a}}}{\mu_\mathrm{h} + l_{m\mathrm{h}} - p\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\, p,\, \mathrm{m_j}}}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\, \mathrm{m_a})\, c_p\right]_{\theta,\, \mathrm{m_j},\, \mathrm{e}} = -\,\theta\left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\, \mathrm{m_a}} + \left[\frac{\partial}{\partial \mathrm{m_h}}(\Sigma\, \mathrm{m_a})\, c_p\right]_{\theta,\, p,\, \mathrm{m_j}} \frac{\theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\, \mathrm{m_a}} + p\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\, \mathrm{m_a}}}{\mu_\mathrm{h} + l_{m\mathrm{h}} - p\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\, p,\, \mathrm{m_j}}}$$

$$\left(\frac{\partial\, l_{m\mathrm{k}}}{\partial p}\right)_{\theta,\, \mathrm{m_j},\, \mathrm{e}} = -\,\theta\left[\frac{\partial}{\partial \mathrm{m_k}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\, \mathrm{m_a}}\right]_{\theta,\, p,\, \mathrm{m_i}} + \left(\frac{\partial\, l_{m\mathrm{k}}}{\partial \mathrm{m_h}}\right)_{\theta,\, p,\, \mathrm{m_j}} \frac{\theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\, \mathrm{m_a}} + p\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\, \mathrm{m_a}}}{\mu_\mathrm{h} + l_{m\mathrm{h}} - p\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\, p,\, \mathrm{m_j}}}$$

$$\left(\frac{\partial \mu_\mathrm{k}}{\partial p}\right)_{\theta,\, \mathrm{m_j},\, \mathrm{e}} = \left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\, p,\, \mathrm{m_i}} + \left(\frac{\partial \mu_\mathrm{k}}{\partial \mathrm{m_h}}\right)_{\theta,\, p,\, \mathrm{m_j}} \frac{\theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\, \mathrm{m_a}} + p\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\, \mathrm{m_a}}}{\mu_\mathrm{h} + l_{m\mathrm{h}} - p\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\, p,\, \mathrm{m_j}}}$$

* $\mathrm{m_a}$ denotes all the component masses; $\mathrm{m_j}$ all except $\mathrm{m_h}$; $\mathrm{m_i}$ all except $\mathrm{m_k}$.

*Group 12**

θ, $\mathbf{m}_j$, $\mathbf{n}$ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,\mathbf{m}_j,\,\mathbf{n}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,\mathbf{m}_a} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_j} \frac{\theta\left(\dfrac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}{l_{m_h}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{\theta,\,\mathbf{m}_j,\,\mathbf{n}} = \left(\frac{\partial^2 \mathbf{v}}{\partial p^2}\right)_{\theta,\,\mathbf{m}_a} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_j} \frac{\theta\left(\dfrac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}{l_{m_h}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\theta,\,\mathbf{m}_j,\,\mathbf{n}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\theta,\,\mathbf{m}_a} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\theta,\,p,\,\mathbf{m}_j} \frac{\theta\left(\dfrac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}{l_{m_h}}$$

$$\left[\frac{\partial}{\partial p}\left(\Sigma\,\mathbf{m}_a\right) c_p\right]_{\theta,\,\mathbf{m}_j,\,\mathbf{n}} = -\,\theta\left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\,\mathbf{m}_a} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\Sigma\,\mathbf{m}_a\right) c_p\right]_{\theta,\,p,\,\mathbf{m}_j} \frac{\theta\left(\dfrac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}{l_{m_h}}$$

$$\left[\frac{\partial\, l_{m_k}}{\partial p}\right]_{\theta,\,\mathbf{m}_j,\,\mathbf{n}} = -\,\theta\left[\frac{\partial}{\partial \mathbf{m}_k}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_i} + \left(\frac{\partial\, l_{m_k}}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathbf{m}_j} \frac{\theta\left(\dfrac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}{l_{m_h}}$$

$$\left[\frac{\partial \mu_k}{\partial p}\right]_{\theta,\,\mathbf{m}_j,\,\mathbf{n}} = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i} + \left(\frac{\partial \mu_k}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathbf{m}_j} \frac{\theta\left(\dfrac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}}{l_{m_h}}$$

*Group 13**

θ, $\mathbf{m}_j$, ζ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m}_a}\right]_{\theta,\,\mathrm{m}_j,\,\mathrm{z}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m}_a}\right]_{\theta,\,\mathrm{m}_a} - \left[\frac{\partial}{\partial \mathrm{m}_h}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m}_a}\right]_{\theta,\,p,\,\mathrm{m}_j} \frac{\mathrm{v}}{\mu_h}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m}_a}\right]_{\theta,\,\mathrm{m}_j,\,\mathrm{z}} = \left(\frac{\partial^2 \mathrm{v}}{\partial p^2}\right)_{\theta,\,\mathrm{m}_a} - \left[\frac{\partial}{\partial \mathrm{m}_h}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m}_a}\right]_{\theta,\,p,\,\mathrm{m}_j} \frac{\mathrm{v}}{\mu_h}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}\right)_{\theta,\,p,\,\mathrm{m}_i}\right]_{\theta,\,\mathrm{m}_j,\,\mathrm{z}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}\right)_{\theta,\,p,\,\mathrm{m}_i}\right]_{\theta,\,\mathrm{m}_a} - \left[\frac{\partial}{\partial \mathrm{m}_h}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}\right)_{\theta,\,p,\,\mathrm{m}_i}\right]_{\theta,\,p,\,\mathrm{m}_j} \frac{\mathrm{v}}{\mu_h}$$

$$\left[\frac{\partial}{\partial p}\,(\Sigma\,\mathrm{m}_a)\,c_p\right]_{\theta,\,\mathrm{m}_j,\,\mathrm{z}} = -\,\theta\left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m}_a} - \left[\frac{\partial}{\partial \mathrm{m}_h}\,(\Sigma\,\mathrm{m}_a)\,c_p\right]_{\theta,\,p,\,\mathrm{m}_j} \frac{\mathrm{v}}{\mu_h}$$

$$\left[\frac{\partial\, l_{mk}}{\partial p}\right]_{\theta,\,\mathrm{m}_j,\,\mathrm{z}} = -\,\theta\left[\frac{\partial}{\partial \mathrm{m}_k}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m}_a}\right]_{\theta,\,p,\,\mathrm{m}_i} - \left(\frac{\partial\, l_{mk}}{\partial \mathrm{m}_h}\right)_{\theta,\,p,\,\mathrm{m}_j} \frac{\mathrm{v}}{\mu_h}$$

$$\left[\frac{\partial \mu_k}{\partial p}\right]_{\theta,\,\mathrm{m}_j,\,\mathrm{z}} = \left(\frac{\partial \mathrm{v}}{\partial \mathrm{m}_k}\right)_{\theta,\,p,\,\mathrm{m}_i} - \left(\frac{\partial \mu_k}{\partial \mathrm{m}_h}\right)_{\theta,\,p,\,\mathrm{m}_j} \frac{\mathrm{v}}{\mu_h}$$

* $\mathbf{m}_a$ denotes all the component masses; $\mathbf{m}_j$ all except $\mathbf{m}_h$; $\mathbf{m}_i$ all except $\mathbf{m}_k$.

*Group 14**

θ, $\mathbf{m}_j$, χ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\, \mathbf{m}_a}\right]_{\theta,\, \mathbf{m}_j,\, \chi} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\, \mathbf{m}_a}\right]_{\theta,\, \mathbf{m}_a} - \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\, \mathbf{m}_a}\right]_{\theta,\, p,\, \mathbf{m}_j} \frac{\mathbf{v} - \theta\left(\dfrac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\, \mathbf{m}_a}}{\mu_h + l_{m_h}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\, \mathbf{m}_a}\right]_{\theta,\, \mathbf{m}_j,\, \chi} = \left(\frac{\partial^2 \mathbf{v}}{\partial p^2}\right)_{\theta,\, \mathbf{m}_a} - \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\, \mathbf{m}_a}\right]_{\theta,\, p,\, \mathbf{m}_j} \frac{\mathbf{v} - \theta\left(\dfrac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\, \mathbf{m}_a}}{\mu_h + l_{m_h}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\, p,\, \mathbf{m}_i}\right]_{\theta,\, \mathbf{m}_j,\, \chi} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\, p,\, \mathbf{m}_i}\right]_{\theta,\, \mathbf{m}_a} - \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\, p,\, \mathbf{m}_i}\right]_{\theta,\, p,\, \mathbf{m}_j} \frac{\mathbf{v} - \theta\left(\dfrac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\, \mathbf{m}_a}}{\mu_h + l_{m_h}}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\, \mathbf{m}_a)\, c_p\right]_{\theta,\, \mathbf{m}_j,\, \chi} = -\,\theta\left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\, \mathbf{m}_a} - \left[\frac{\partial}{\partial \mathbf{m}_h}(\Sigma\, \mathbf{m}_a)\, c_p\right]_{\theta,\, p,\, \mathbf{m}_j} \frac{\mathbf{v} - \theta\left(\dfrac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\, \mathbf{m}_a}}{\mu_h + l_{m_h}}$$

$$\left[\frac{\partial\, l_{m_k}}{\partial p}\right]_{\theta,\, \mathbf{m}_j,\, \chi} = -\,\theta\left[\frac{\partial}{\partial \mathbf{m}_k}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\, \mathbf{m}_a}\right]_{\theta,\, p,\, \mathbf{m}_i} - \left(\frac{\partial\, l_{m_k}}{\partial \mathbf{m}_h}\right)_{\theta,\, p,\, \mathbf{m}_j} \frac{\mathbf{v} - \theta\left(\dfrac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\, \mathbf{m}_a}}{\mu_h + l_{m_h}}$$

$$\left[\frac{\partial \mu_k}{\partial p}\right]_{\theta,\, \mathbf{m}_j,\, \chi} = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\, p,\, \mathbf{m}_i} - \left(\frac{\partial \mu_k}{\partial \mathbf{m}_h}\right)_{\theta,\, p,\, \mathbf{m}_j} \frac{\mathbf{v} - \theta\left(\dfrac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\, \mathbf{m}_a}}{\mu_h + l_{m_h}}$$

*Group 15**

θ, $\mathrm{m_j}$, ψ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,\mathrm{m_j},\,\mathrm{y}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{p\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}}{\mu_{\mathrm{h}} - p\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\theta,\,\mathrm{m_j},\,\mathrm{y}} = \left(\frac{\partial^2 \mathrm{v}}{\partial p^2}\right)_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{p\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}}{\mu_{\mathrm{h}} - p\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,\mathrm{m_j},\,\mathrm{y}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,p,\,\mathrm{m_j}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{p\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}}{\mu_{\mathrm{h}} - p\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,\mathrm{m_j},\,\mathrm{y}} = -\,\theta\left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \mathrm{m_h}}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{p\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}}{\mu_{\mathrm{h}} - p\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta.\,p,\,\mathrm{m_j}}}$$

* $\mathrm{m_a}$ denotes all the component masses; $\mathrm{m_j}$ all except $\mathrm{m_h}$; $\mathrm{m_i}$ all except $\mathrm{m_k}$.

*Group 15** (Con.)

$$\left(\frac{\partial l_{m\mathrm{k}}}{\partial p}\right)_{\theta,\,\mathrm{m_j},\,\mathrm{y}} = -\theta\left[\frac{\partial}{\partial \mathrm{m_k}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_i}} + \left(\frac{\partial l_{m\mathrm{k}}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}} \frac{p\left(\dfrac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}}{\mu_\mathrm{h} - p\left(\dfrac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left(\frac{\partial \mu_\mathrm{k}}{\partial p}\right)_{\theta,\,\mathrm{m_j},\,\mathrm{y}} = \left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}} + \left(\frac{\partial \mu_\mathrm{k}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}} \frac{p\left(\dfrac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}}{\mu_\mathrm{h} - p\left(\dfrac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

*Group 16**

p, $\mathrm{m_j}$, v constant.

$$\left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{p,\,\mathrm{m_j},\,\mathrm{v}} = \left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}} - \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{\left(\dfrac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}}{\left(\dfrac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{p,\,\mathrm{m_j},\,\mathrm{v}} = \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{p,\,\mathrm{m_a}} - \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{\left(\dfrac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}}{\left(\dfrac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{p,\,\mathrm{m_j},\,\mathrm{v}} = \left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{p,\,\mathrm{m_a}} - \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{\left(\frac{\partial \mathrm{v}}{\partial\theta}\right)_{p,\,\mathrm{m_a}}}{\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial}{\partial\theta}\,(\Sigma\,\mathrm{m_a})\,c_p\right]_{p,\,\mathrm{m_j},\,\mathrm{v}} = (\Sigma\,\mathrm{m_a})\left(\frac{\partial\,c_p}{\partial\theta}\right)_{p,\,\mathrm{m_a}} - \left[\frac{\partial}{\partial \mathrm{m_h}}\,(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{\left(\frac{\partial \mathrm{v}}{\partial\theta}\right)_{p,\,\mathrm{m_a}}}{\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial\,l_{m_k}}{\partial\theta}\right]_{p,\,\mathrm{m_j},\,\mathrm{v}} = \frac{l_{m_k}}{\theta} + \left[\frac{\partial}{\partial \mathrm{m_k}}\,(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_i}} - \left(\frac{\partial\,l_{m_k}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}} \frac{\left(\frac{\partial \mathrm{v}}{\partial\theta}\right)_{p,\,\mathrm{m_a}}}{\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial\mu_{\mathrm{k}}}{\partial\theta}\right]_{p,\,\mathrm{m_j},\,\mathrm{v}} = -\frac{l_{m_k}}{\theta} - \left(\frac{\partial\mu_{\mathrm{k}}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}} \frac{\left(\frac{\partial \mathrm{v}}{\partial\theta}\right)_{p,\,\mathrm{m_a}}}{\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

* $\mathrm{m_a}$ denotes all the component masses; $\mathrm{m_j}$ all except $\mathrm{m_h}$; $\mathrm{m_i}$ all except $\mathrm{m_k}$.

*Group 17**

p, $\mathbf{m}_j$, $\boldsymbol{\varepsilon}$ constant.

$$\left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathrm{v}}{\partial\theta}\right)_{p,\,\mathrm{m_a}}\right]_{p,\,\mathrm{m_j},\,\mathrm{e}} = \left(\frac{\partial^2 \mathrm{v}}{\partial\theta^2}\right)_{p,\,\mathrm{m_a}} - \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial\theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{(\Sigma\,\mathrm{m_a})\,c_p - p\left(\dfrac{\partial \mathrm{v}}{\partial\theta}\right)_{p,\,\mathrm{m_a}}}{\mu_\mathrm{h} + l_{m_\mathrm{h}} - p\left(\dfrac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{p,\,\mathrm{m_j},\,\mathrm{e}} = \left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{p,\,\mathrm{m_a}} - \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{(\Sigma\,\mathrm{m_a})\,c_p - p\left(\dfrac{\partial \mathrm{v}}{\partial\theta}\right)_{p,\,\mathrm{m_a}}}{\mu_\mathrm{h} + l_{m_\mathrm{h}} - p\left(\dfrac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{p,\,\mathrm{m_j},\,\mathrm{e}} = \left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{p,\,\mathrm{m_a}} - \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{(\Sigma\,\mathrm{m_a})\,c_p - p\left(\dfrac{\partial \mathrm{v}}{\partial\theta}\right)_{p,\,\mathrm{m_a}}}{\mu_\mathrm{h} + l_{m_\mathrm{h}} - p\left(\dfrac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial}{\partial\theta}(\Sigma\,\mathrm{m_a})\,c_p\right]_{p,\,\mathrm{m_j},\,\mathrm{e}} = (\Sigma\,\mathrm{m_a})\left(\frac{\partial c_p}{\partial\theta}\right)_{p,\,\mathrm{m_a}} - \left[\frac{\partial}{\partial \mathrm{m_h}}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{(\Sigma\,\mathrm{m_a})\,c_p - p\left(\dfrac{\partial \mathrm{v}}{\partial\theta}\right)_{p,\,\mathrm{m_a}}}{\mu_\mathrm{h} + l_{m_\mathrm{h}} - p\left(\dfrac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial\, l_{m_k}}{\partial\theta}\right]_{p,\,\mathbf{m}_j,\,\mathbf{e}} = \frac{l_{m_k}}{\theta} + \left[\frac{\partial}{\partial \mathbf{m}_k}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{\theta,\,p,\,\mathbf{m}_i} - \left(\frac{\partial\, l_{m_k}}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathbf{m}_j} \frac{(\Sigma\,\mathbf{m}_a)\,c_p - p\left(\dfrac{\partial \mathbf{v}}{\partial\theta}\right)_{p,\,\mathbf{m}_a}}{\mu_h + l_{m_h} - p\left(\dfrac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathbf{m}_j}}$$

$$\left[\frac{\partial \mu_k}{\partial\theta}\right]_{p,\,\mathbf{m}_j,\,\mathbf{e}} = -\frac{l_{m_k}}{\theta} - \left(\frac{\partial \mu_k}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathbf{m}_j} \frac{(\Sigma\,\mathbf{m}_a)\,c_p - p\left(\dfrac{\partial \mathbf{v}}{\partial\theta}\right)_{p,\,\mathbf{m}_a}}{\mu_h + l_{m_h} - p\left(\dfrac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathbf{m}_j}}$$

*Group 18**

p, $\mathbf{m}_j$, $\mathbf{n}$ constant.

$$\left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathbf{v}}{\partial\theta}\right)_{p,\,\mathbf{m}_a}\right]_{p,\,\mathbf{m}_j,\,\mathbf{n}} = \left(\frac{\partial^2 \mathbf{v}}{\partial\theta^2}\right)_{p,\,\mathbf{m}_a} - \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial\theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_j} \frac{(\Sigma\,\mathbf{m}_a)\,c_p}{l_{m_h}}$$

$$\left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{p,\,\mathbf{m}_j,\,\mathbf{n}} = \left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{p,\,\mathbf{m}_a} - \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_j} \frac{(\Sigma\,\mathbf{m}_a)\,c_p}{l_{m_h}}$$

* $\mathbf{m}_a$ denotes all the component masses; $\mathbf{m}_j$ all except $\mathbf{m}_h$; $\mathbf{m}_i$ all except $\mathbf{m}_k$.

*Group 18** (Con.)

$$\left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{p,\,\mathbf{m}_j,\,\mathbf{n}} = \left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{p,\,\mathbf{m}_a} - \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\theta,\,p,\,\mathbf{m}_j} \frac{(\Sigma\,\mathbf{m}_a)\,c_p}{l_{m_h}}$$

$$\left[\frac{\partial}{\partial\theta}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{p,\,\mathbf{m}_j,\,\mathbf{n}} = \left[\frac{\partial}{\partial\theta}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{p,\,\mathbf{m}_a} - \left[\frac{\partial}{\partial \mathbf{m}_h}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{\theta,\,p,\,\mathbf{m}_j} \frac{(\Sigma\,\mathbf{m}_a)\,c_p}{l_{m_h}}$$

$$\left[\frac{\partial\, l_{m_k}}{\partial\theta}\right]_{p,\,\mathbf{m}_j,\,\mathbf{n}} = \frac{l_{m_k}}{\theta} + \left[\frac{\partial}{\partial \mathbf{m}_k}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{\theta,\,p,\,\mathbf{m}_i} - \left(\frac{\partial\, l_{m_k}}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathbf{m}_j} \frac{(\Sigma\,\mathbf{m}_a)\,c_p}{l_{m_h}}$$

$$\left[\frac{\partial\mu_k}{\partial\theta}\right]_{p,\,\mathbf{m}_j,\,\mathbf{n}} = -\frac{l_{m_k}}{\theta} - \left(\frac{\partial\mu_k}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathbf{m}_j} \frac{(\Sigma\,\mathbf{m}_a)\,c_p}{l_{m_h}}$$

*Group 19**

p, $\mathbf{m}_j$, ζ constant.

$$\left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathbf{v}}{\partial\theta}\right)_{p,\,\mathbf{m}_a}\right]_{p,\,\mathbf{m}_j,\,\mathbf{z}} = \left(\frac{\partial^2 \mathbf{v}}{\partial\theta^2}\right)_{p,\,\mathbf{m}_a} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial\theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_j} \frac{\mathbf{n}}{\mu_h}$$

$$\left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{p,\,\mathbf{m}_j,\,\mathbf{z}} = \left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{p,\,\mathbf{m}_a} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_j} \frac{\mathbf{n}}{\mu_h}$$

$$\left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{p,\,\mathbf{m}_j,\,\mathbf{z}} = \left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{p,\,\mathbf{m}_a} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\theta,\,p,\,\mathbf{m}_j} \frac{\mathbf{n}}{\mu_h}$$

$$\left[\frac{\partial}{\partial\theta}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{p,\,\mathbf{m}_j,\,\mathrm{z}} = (\Sigma\,\mathbf{m}_a)\left(\frac{\partial c_p}{\partial\theta}\right)_{p,\,\mathbf{m}_a} + \left[\frac{\partial}{\partial\mathbf{m}_h}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{\theta,\,p,\,\mathbf{m}_j}\frac{\mathbf{n}}{\mu_h}$$

$$\left[\frac{\partial l_{m_k}}{\partial\theta}\right]_{p,\,\mathbf{m}_j,\,\mathrm{z}} = \frac{l_{m_k}}{\theta} + \left[\frac{\partial}{\partial\mathbf{m}_k}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{\theta,\,p,\,\mathbf{m}_i} + \left[\frac{\partial l_{m_k}}{\partial\mathbf{m}_h}\right]_{\theta,\,p,\,\mathbf{m}_j}\frac{\mathbf{n}}{\mu_h}$$

$$\left[\frac{\partial\mu_k}{\partial\theta}\right]_{p,\,\mathbf{m}_j,\,\mathrm{z}} = -\frac{l_{m_k}}{\theta} + \left(\frac{\partial\mu_k}{\partial\mathbf{m}_h}\right)_{\theta,\,p,\,\mathbf{m}_j}\frac{\mathbf{n}}{\mu_h}$$

*Group 20**

p, $\mathbf{m}_j$, χ constant.

$$\left[\frac{\partial}{\partial\theta}\left(\frac{\partial\mathrm{v}}{\partial\theta}\right)_{p,\,\mathbf{m}_a}\right]_{p,\,\mathbf{m}_j,\,\mathrm{x}} = \left(\frac{\partial^2\mathrm{v}}{\partial\theta^2}\right)_{p,\,\mathbf{m}_a} - \left[\frac{\partial}{\partial\mathbf{m}_h}\left(\frac{\partial\mathrm{v}}{\partial\theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_j}\frac{(\Sigma\,\mathbf{m}_a)\,c_p}{\mu_h + l_{m_h}}$$

$$\left[\frac{\partial}{\partial\theta}\left(\frac{\partial\mathrm{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{p,\,\mathbf{m}_j,\,\mathrm{x}} = \left[\frac{\partial}{\partial\theta}\left(\frac{\partial\mathrm{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{p,\,\mathbf{m}_a} - \left[\frac{\partial}{\partial\mathbf{m}_h}\left(\frac{\partial\mathrm{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_j}\frac{(\Sigma\,\mathbf{m}_a)\,c_p}{\mu_h + l_{m_h}}$$

$$\left[\frac{\partial}{\partial\theta}\left(\frac{\partial\mathrm{v}}{\partial\mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{p,\,\mathbf{m}_j,\,\mathrm{x}} = \left[\frac{\partial}{\partial\theta}\left(\frac{\partial\mathrm{v}}{\partial\mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{p,\,\mathbf{m}_a} - \left[\frac{\partial}{\partial\mathbf{m}_h}\left(\frac{\partial\mathrm{v}}{\partial\mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\theta,\,p,\,\mathbf{m}_j}\frac{(\Sigma\,\mathbf{m}_a)\,c_p}{\mu_h + l_{m_h}}$$

$$\left[\frac{\partial}{\partial\theta}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{p,\,\mathbf{m}_j,\,\mathrm{x}} = (\Sigma\,\mathbf{m}_a)\left(\frac{\partial c_p}{\partial\theta}\right)_{p,\,\mathbf{m}_a} - \left[\frac{\partial}{\partial\mathbf{m}_h}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{\theta,\,p,\,\mathbf{m}_j}\frac{(\Sigma\,\mathbf{m}_a)\,c_p}{\mu_h + l_{m_h}}$$

* $\mathbf{m}_a$ denotes all the component masses; $\mathbf{m}_j$ all except $\mathbf{m}_h$; $\mathbf{m}_i$ all except $\mathbf{m}_k$.

*Group 20** (Con.)

$$\left[\frac{\partial\, l_{m_k}}{\partial\theta}\right]_{p,\,\mathrm{m_j},\,\mathrm{x}} = \frac{l_{m_k}}{\theta} + \left[\frac{\partial}{\partial \mathbf{m}_k}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{\theta,\,p,\,\mathrm{m_i}} - \left(\frac{\partial\, l_{m_k}}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathrm{m_j}} \frac{(\Sigma\,\mathbf{m}_a)\,c_p}{\mu_h + l_{m_h}}$$

$$\left[\frac{\partial\mu_k}{\partial\theta}\right]_{p,\,\mathrm{m_j},\,\mathrm{x}} = -\frac{l_{m_k}}{\theta} - \left(\frac{\partial\mu_k}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathrm{m_j}} \frac{(\Sigma\,\mathbf{m}_a)\,c_p}{\mu_h + l_{m_h}}$$

*Group 21**

p, $\mathbf{m}_j$, ψ constant.

$$\left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathrm{v}}{\partial\theta}\right)_{p,\,\mathrm{m_a}}\right]_{p,\,\mathrm{m_j},\,\mathrm{y}} = \left(\frac{\partial^2 \mathrm{v}}{\partial\theta^2}\right)_{p,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathrm{v}}{\partial\theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{\mathfrak{n} + p\left(\frac{\partial \mathrm{v}}{\partial\theta}\right)_{p,\,\mathrm{m_a}}}{\mu_h - p\left(\frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{p,\,\mathrm{m_j},\,\mathrm{y}} = \left[\frac{\partial}{\partial\theta}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{p,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{\mathfrak{n} + p\left(\frac{\partial \mathrm{v}}{\partial\theta}\right)_{p,\,\mathrm{m_a}}}{\mu_h - p\left(\frac{\partial \mathrm{v}}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{p,\,\mathrm{m_j},\,\mathrm{y}} = \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{p,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{\mathbf{n} + p\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}}{\mu_\mathrm{h} - p\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial}{\partial \theta}(\Sigma\, \mathrm{m_a})\, c_p\right]_{p,\,\mathrm{m_j},\,\mathrm{y}} = (\Sigma\, \mathrm{m_a})\left(\frac{\partial c_p}{\partial \theta}\right)_{p,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial c_p}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,p,\,\mathrm{m_j}} \frac{\mathbf{n} + p\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}}{\mu_\mathrm{h} - p\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial\, l_{m\mathrm{k}}}{\partial \theta}\right]_{p,\,\mathrm{m_j},\,\mathrm{y}} = \frac{l_{m\mathrm{k}}}{\theta} + \left[\frac{\partial}{\partial \mathrm{m_k}}(\Sigma\, \mathrm{m_a})\, c_p\right]_{\theta,\,p,\,\mathrm{m_i}} + \left(\frac{\partial\, l_{m\mathrm{k}}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}} \frac{\mathbf{n} + p\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}}{\mu_\mathrm{h} - p\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

$$\left[\frac{\partial \mu_\mathrm{k}}{\partial \theta}\right]_{p,\,\mathrm{m_j},\,\mathrm{y}} = -\frac{l_{m\mathrm{k}}}{\theta} + \left(\frac{\partial \mu_\mathrm{k}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}} \frac{\mathbf{n} + p\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}}{\mu_\mathrm{h} - p\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}}$$

* $\mathrm{m_a}$ denotes all the component masses; $\mathrm{m_j}$ all except $\mathrm{m_h}$; $\mathrm{m_i}$ all except $\mathrm{m_k}$.

*Group 22**

$\mathbf{m}_j$, v, ε constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\mathrm{m_j,\,v,\,e}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,\mathrm{m_a}} + \left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)^{\mathrm{A}}_{\mathrm{m_j,\,v,\,e}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)^{\mathrm{B}}_{\mathrm{m_j,\,v,\,e}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\mathrm{m_j,\,v,\,e}} = \left(\frac{\partial^2 \mathrm{v}}{\partial p^2}\right)_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,v,\,e}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,v,\,e}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\mathrm{m_j,\,v,\,e}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,v,\,e}} +$$
$$\left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,v,\,e}}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\mathrm{m_j,\,v,\,e}} = -\,\theta\left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}} + (\Sigma\,\mathrm{m_a})\left(\frac{\partial\, c_p}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,v,\,e}} + \left[\frac{\partial}{\partial \mathrm{m_h}}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,v,\,e}}$$

$$\left[\frac{\partial\, l_{m\mathrm{k}}}{\partial p}\right]_{\mathrm{m_j,\,v,\,e}} = -\,\theta\left[\frac{\partial}{\partial \mathrm{m_k}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_i}} + \left\{\frac{l_{m\mathrm{k}}}{\theta} + \left[\frac{\partial}{\partial \mathrm{m_k}}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_i}}\right\}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,v,\,e}} + \left(\frac{\partial\, l_{m\mathrm{k}}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,v,\,e}}$$

$$\left[\frac{\partial \mu_k}{\partial p}\right]_{m_j,\,v,\,e} = \left(\frac{\partial v}{\partial m_k}\right)_{\theta,\,p,\,m_i} - \frac{l_{m_k}}{\theta}\left(\frac{\partial \theta}{\partial p}\right)_{m_j,\,v,\,e} + \left(\frac{\partial \mu_k}{\partial m_h}\right)_{\theta,\,p,\,m_j}\left(\frac{\partial m_h}{\partial p}\right)_{m_j,\,v,\,e}$$

---

where

$$\left(\frac{\partial \theta}{\partial p}\right)_{m_j,\,v,\,e} = \frac{-(\mu_h + l_{m_h})\left(\frac{\partial v}{\partial p}\right)_{\theta,\,m_a} - \theta\left(\frac{\partial v}{\partial \theta}\right)_{p,\,m_a}\left(\frac{\partial v}{\partial m_h}\right)_{\theta,\,p,\,m_j}}{(\mu_h + l_{m_h})\left(\frac{\partial v}{\partial \theta}\right)_{p,\,m_a} - (\Sigma\, m_a)\, c_p\left(\frac{\partial v}{\partial m_h}\right)_{\theta,\,p,\,m_j}}$$

and

$$\left(\frac{\partial m_h}{\partial p}\right)_{m_j,\,v,\,e} = \frac{(\Sigma\, m_a)\, c_p\left(\frac{\partial v}{\partial p}\right)_{\theta,\,m_a} + \theta\left(\frac{\partial v}{\partial \theta}\right)^2_{p,\,m_a}}{(\mu_h + l_{m_h})\left(\frac{\partial v}{\partial \theta}\right)_{p,\,m_a} - (\Sigma\, m_a)\, c_p\left(\frac{\partial v}{\partial m_h}\right)_{\theta,\,p,\,m_j}}$$

* m_a denotes all the component masses; m_j all except m_h; m_i all except m_k.

*Group 23**

$\mathbf{m}_j$, $\mathbf{v}$, $\mathbf{n}$ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\mathrm{m_j,\,v,\,n}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,\mathrm{m_a}} + \left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,v,\,n}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,v,\,n}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\mathrm{m_j,\,v,\,n}} = \left(\frac{\partial^2 \mathrm{v}}{\partial p^2}\right)_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,v,\,n}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,v,\,n}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\mathrm{m_j,\,v,\,n}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,v,\,n}} +$$
$$\left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,v,\,n}}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\mathrm{m_j,\,v,\,n}} = -\,\theta\left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}} + (\Sigma\,\mathrm{m_a})\left(\frac{\partial c_p}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,v,\,n}} + \left[\frac{\partial}{\partial \mathrm{m_h}}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,v,\,n}}$$

$$\left[\frac{\partial\, l_{m\mathrm{k}}}{\partial p}\right]_{\mathrm{m_j,\,v,\,n}} = -\,\theta\left[\frac{\partial}{\partial \mathrm{m_k}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_i}} + \left\{\frac{l_{m\mathrm{k}}}{\theta} + \left[\frac{\partial}{\partial \mathrm{m_k}}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_i}}\right\}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,v,\,n}} + \left(\frac{\partial\, l_{m\mathrm{k}}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,v,\,n}}$$

$$\left(\frac{\partial \mu_{\mathrm{k}}}{\partial p}\right)_{\mathrm{m_j, v, n}} = \left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta, p, \mathrm{m_i}} - \frac{l_{m_{\mathrm{k}}}}{\theta}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j, v, n}} + \left(\frac{\partial \mu_{\mathrm{k}}}{\partial \mathrm{m_h}}\right)_{\theta, p, \mathrm{m_j}} \left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j, v, n}}$$

---

where

$$\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j, v, n}} = \frac{- l_{m_{\mathrm{h}}}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta, \mathrm{m_a}} - \theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p, \mathrm{m_a}}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta, p, \mathrm{m_j}}}{+ l_{m_{\mathrm{h}}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p, \mathrm{m_a}} - (\Sigma\, \mathrm{m_a})\, c_p \left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta, p, \mathrm{m_j}}}$$

and

$$\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j, v, n}} = \frac{(\Sigma\, \mathrm{m_a})\, c_p \left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta, \mathrm{m_a}} + \theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2_{p, \mathrm{m_a}}}{l_{m_{\mathrm{h}}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p, \mathrm{m_a}} - (\Sigma\, \mathrm{m_a})\, c_p \left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta, p, \mathrm{m_j}}}$$

* $\mathbf{m_a}$ denotes all the component masses; $\mathbf{m_j}$ all except $\mathbf{m_h}$; $\mathbf{m_i}$ all except $\mathbf{m_k}$.

*Group 24**

$\mathrm{m_j}$, v, ζ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\mathrm{m_j,\,v,\,z}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,\mathrm{m_a}} + \left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,v,\,z}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,v,\,z}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\mathrm{m_j,\,v,\,z}} = \left(\frac{\partial^2 \mathrm{v}}{\partial p^2}\right)_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,v,\,z}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,v,\,z}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\mathrm{m_j,\,v,\,z}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,v,\,z}} +$$
$$\left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,v,\,z}}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\mathrm{m_j,\,v,\,z}} = -\,\theta\left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}} + (\Sigma\,\mathrm{m_a})\left(\frac{\partial\, c_p}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,v,\,z}} + \left[\frac{\partial}{\partial \mathrm{m_h}}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,v,\,z}}$$

$$\left[\frac{\partial\, l_{m_k}}{\partial p}\right]_{\mathrm{m_j,\,v,\,z}} = -\,\theta\left[\frac{\partial}{\partial \mathrm{m_k}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_i}} + \left\{\frac{l_{m_k}}{\theta} + \left[\frac{\partial}{\partial \mathrm{m_k}}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_i}}\right\}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,v,\,z}} + \left(\frac{\partial\, l_{m_k}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,v,\,z}}$$

$$\left[\frac{\partial \mu_k}{\partial p}\right]_{m_j, v, z} = \left(\frac{\partial v}{\partial m_k}\right)_{\theta, p, m_i} - \frac{l_{m_k}}{\theta}\left(\frac{\partial \theta}{\partial p}\right)_{m_j, v, z} + \left(\frac{\partial \mu_k}{\partial m_h}\right)_{\theta, p, m_j}\left(\frac{\partial m_h}{\partial p}\right)_{m_j, v, z}$$

---

where

$$\left(\frac{\partial \theta}{\partial p}\right)_{m_j, v, z} = \frac{-\mu_h\left(\frac{\partial v}{\partial p}\right)_{\theta, m_a} + v\left(\frac{\partial v}{\partial m_h}\right)_{\theta, p, m_j}}{\mu_h\left(\frac{\partial v}{\partial \theta}\right)_{p, m_a} + \mathfrak{n}\left(\frac{\partial v}{\partial m_h}\right)_{\theta, p, m_j}}$$

and

$$\left(\frac{\partial m_h}{\partial p}\right)_{m_j, v, z} = \frac{-v\left(\frac{\partial v}{\partial \theta}\right)_{p, m_a} - \mathfrak{n}\left(\frac{\partial v}{\partial p}\right)_{\theta, m_a}}{\mu_h\left(\frac{\partial v}{\partial \theta}\right)_{p, m_a} + \mathfrak{n}\left(\frac{\partial v}{\partial m_h}\right)_{\theta, p, m_j}}$$

* m_a denotes all the component masses; m_j all except m_h; m_i all except m_k.

*Group 25**

m_j, v, χ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right]_{m_j,\, v,\, x} = \left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right]_{\theta,\, m_a} + \left(\frac{\partial^2 v}{\partial \theta^2}\right)_{p,\, m_a}\left(\frac{\partial \theta}{\partial p}\right)_{m_j,\, v,\, x} + \left[\frac{\partial}{\partial m_h}\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right]_{\theta,\, p,\, m_j}\left(\frac{\partial m_h}{\partial p}\right)_{m_j,\, v,\, x}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_a}\right]_{m_j,\, v,\, x} = \left(\frac{\partial^2 v}{\partial p^2}\right)_{\theta,\, m_a} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_a}\right]_{p,\, m_a}\left(\frac{\partial \theta}{\partial p}\right)_{m_j,\, v,\, x} + \left[\frac{\partial}{\partial m_h}\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_a}\right]_{\theta,\, p,\, m_j}\left(\frac{\partial m_h}{\partial p}\right)_{m_j,\, v,\, x}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial m_k}\right)_{\theta,\, p,\, m_i}\right]_{m_j,\, v,\, x} = \left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial m_k}\right)_{\theta,\, p,\, m_i}\right]_{\theta,\, m_a} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial m_k}\right)_{\theta,\, p,\, m_i}\right]_{p,\, m_a}\left(\frac{\partial \theta}{\partial p}\right)_{m_j,\, v,\, x} +$$
$$\left[\frac{\partial}{\partial m_h}\left(\frac{\partial v}{\partial m_k}\right)_{\theta,\, p,\, m_i}\right]_{\theta,\, p,\, m_j}\left(\frac{\partial m_h}{\partial p}\right)_{m_j,\, v,\, x}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\, m_a)\, c_p\right]_{m_j,\, v,\, x} = -\,\theta\left(\frac{\partial^2 v}{\partial \theta^2}\right)_{p,\, m_a} + (\Sigma\, m_a)\left(\frac{\partial\, c_p}{\partial \theta}\right)_{p,\, m_a}\left(\frac{\partial \theta}{\partial p}\right)_{m_j,\, v,\, x} + \left[\frac{\partial}{\partial m_h}(\Sigma\, m_a)\, c_p\right]_{\theta,\, p,\, m_j}\left(\frac{\partial m_h}{\partial p}\right)_{m_j,\, v,\, x}$$

$$\left[\frac{\partial\, l_{mk}}{\partial p}\right]_{m_j,\, v,\, x} = -\,\theta\left[\frac{\partial}{\partial m_k}\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right]_{\theta,\, p,\, m_i} + \left\{\frac{l_{mk}}{\theta} + \left[\frac{\partial}{\partial m_k}(\Sigma\, m_a)\, c_p\right]_{\theta,\, p,\, m_i}\right\}\left(\frac{\partial \theta}{\partial p}\right)_{m_j,\, v,\, x} + \left(\frac{\partial\, l_{mk}}{\partial m_h}\right)_{\theta,\, p,\, m_j}\left(\frac{\partial m_h}{\partial p}\right)_{m_j,\, v,\, x}$$

$$\left[\frac{\partial \mu_{\mathrm{k}}}{\partial p}\right]_{\mathrm{m_j, v, x}} = \left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta, p, \mathrm{m_i}} - \frac{l_{m\mathrm{k}}}{\theta}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j, v, x}} + \left(\frac{\partial \mu_{\mathrm{k}}}{\partial \mathrm{m_h}}\right)_{\theta, p, \mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j, v, x}}$$

---

where

$$\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j, v, x}} = \frac{-(\mu_{\mathrm{h}} + l_{m\mathrm{h}})\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta, \mathrm{m_a}} - \left[\theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p, \mathrm{m_a}} - \mathrm{v}\right]\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta, p, \mathrm{m_j}}}{(\mu_{\mathrm{h}} + l_{m\mathrm{h}})\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p, \mathrm{m_a}} - (\Sigma\, \mathrm{m_a})\, c_p\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta, p, \mathrm{m_j}}}$$

and

$$\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j, v, x}} = \frac{-\mathrm{v}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p, \mathrm{m_a}} + (\Sigma\, \mathrm{m_a})\, c_p\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta, \mathrm{m_a}} + \theta\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)^2_{p, \mathrm{m_a}}}{(\mu_{\mathrm{h}} + l_{m\mathrm{h}})\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p, \mathrm{m_a}} - (\Sigma\, \mathrm{m_a})\, c_p\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_h}}\right)_{\theta, p, \mathrm{m_j}}}$$

* $\mathbf{m_a}$ denotes all the component masses; $\mathbf{m_j}$ all except $\mathbf{m_h}$; $\mathbf{m_i}$ all except $\mathbf{m_k}$.

*Group 26**

m_j, v, ψ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right]_{m_j,\, v,\, y} = \left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right]_{\theta,\, m_a} + \left(\frac{\partial^2 v}{\partial \theta^2}\right)_{p,\, m_a}\left(\frac{\partial \theta}{\partial p}\right)_{m_j,\, v,\, y} + \left[\frac{\partial}{\partial m_h}\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right]_{\theta,\, p,\, m_j}\left(\frac{\partial m_h}{\partial p}\right)_{m_j,\, v,\, y}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_a}\right]_{m_j,\, v,\, y} = \left(\frac{\partial^2 v}{\partial p^2}\right)_{\theta,\, m_a} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_a}\right]_{p,\, m_a}\left(\frac{\partial \theta}{\partial p}\right)_{m_j,\, v,\, y} + \left[\frac{\partial}{\partial m_h}\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_a}\right]_{\theta,\, p,\, m_j}\left(\frac{\partial m_h}{\partial p}\right)_{m_j,\, v,\, y}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial m_k}\right)_{\theta,\, p,\, m_i}\right]_{m_j,\, v,\, y} = \left[\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial m_k}\right)_{\theta,\, p,\, m_i}\right]_{\theta,\, m_a} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial m_k}\right)_{\theta,\, p,\, m_i}\right]_{p,\, m_a}\left(\frac{\partial \theta}{\partial p}\right)_{m_j,\, v,\, y} + \left[\frac{\partial}{\partial m_h}\left(\frac{\partial v}{\partial m_k}\right)_{\theta,\, p,\, m_i}\right]_{\theta,\, p,\, m_j}\left(\frac{\partial m_h}{\partial p}\right)_{m_j,\, v,\, y}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\, m_a)\, c_p\right]_{m_j,\, v,\, y} = -\theta\left(\frac{\partial^2 v}{\partial \theta^2}\right)_{p,\, m_a} + (\Sigma\, m_a)\left(\frac{\partial c_p}{\partial \theta}\right)_{p,\, m_a}\left(\frac{\partial \theta}{\partial p}\right)_{m_j,\, v,\, y} + \left[\frac{\partial}{\partial m_h}(\Sigma\, m_a)\, c_p\right]_{\theta,\, p,\, m_j}\left(\frac{\partial m_h}{\partial p}\right)_{m_j,\, v,\, y}$$

$$\left[\frac{\partial\, l_{m_k}}{\partial p}\right]_{m_j,\, v,\, y} = -\theta\left[\frac{\partial}{\partial m_k}\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right]_{\theta,\, p,\, m_i} + \left\{\frac{l_{m_k}}{\theta} + \left[\frac{\partial}{\partial m_k}(\Sigma\, m_a)\, c_p\right]_{\theta,\, p,\, m_i}\right\}\left(\frac{\partial \theta}{\partial p}\right)_{m_j,\, v,\, y} + \left(\frac{\partial\, l_{m_k}}{\partial m_h}\right)_{\theta,\, p,\, m_j}\left(\frac{\partial m_h}{\partial p}\right)_{m_j,\, v,\, y}$$

$$\left[\frac{\partial \mu_k}{\partial p}\right]_{m_j,\,v,\,y} = \left(\frac{\partial v}{\partial m_k}\right)_{\theta,\,p,\,m_i} - \frac{l_{m_k}}{\theta}\left(\frac{\partial \theta}{\partial p}\right)_{m_j,\,v,\,y} + \left(\frac{\partial \mu_k}{\partial m_h}\right)_{\theta,\,p,\,m_j}\left(\frac{\partial m_h}{\partial p}\right)_{m_j,\,v,\,y}$$

---

where

$$\left(\frac{\partial \theta}{\partial p}\right)_{m_j,\,v,\,y} = \frac{-\mu_h\left(\frac{\partial v}{\partial p}\right)_{\theta,\,m_a}}{\mu_h\left(\frac{\partial v}{\partial \theta}\right)_{p,\,m_a} + n\left(\frac{\partial v}{\partial m_h}\right)_{\theta,\,p,\,m_j}}$$

and

$$\left(\frac{\partial m_h}{\partial p}\right)_{m_j,\,v,\,y} = \frac{-n\left(\frac{\partial v}{\partial p}\right)_{\theta,\,m_a}}{\mu_h\left(\frac{\partial v}{\partial \theta}\right)_{p,\,m_a} + n\left(\frac{\partial v}{\partial m_h}\right)_{\theta,\,p,\,m_j}}$$

* m_a denotes all the component masses; m_j all except m_h; m_i all except m_k.

*Group 27**

$\mathbf{m}_j$, $\boldsymbol{\varepsilon}$, $\mathbf{n}$ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{n}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,\mathbf{m}_a} + \left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\,\mathbf{m}_a}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{n}} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{n}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{n}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{\theta,\,\mathbf{m}_a} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{p,\,\mathbf{m}_a}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{n}} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{n}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{n}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\theta,\,\mathbf{m}_a} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{p,\,\mathbf{m}_a}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{n}} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{n}}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{n}} = -\theta\left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\,\mathbf{m}_a} + (\Sigma\,\mathbf{m}_a)\left(\frac{\partial\,c_p}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{n}} + \left[\frac{\partial}{\partial \mathbf{m}_h}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{\theta,\,p,\,\mathbf{m}_i}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{n}}$$

$$\left[\frac{\partial\,l_{m_k}}{\partial p}\right]_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{n}} = -\theta\left[\frac{\partial}{\partial \mathbf{m}_k}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_i} + \left\{\frac{l_{m_k}}{\theta} + \left[\frac{\partial}{\partial \mathbf{m}_k}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{\theta,\,p,\,\mathbf{m}_i}\right\}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{n}} + \left(\frac{\partial\,l_{m_k}}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{n}}$$

$$\left[\frac{\partial \mu_k}{\partial p}\right]_{m_j, e, n} = \left(\frac{\partial v}{\partial m_k}\right)_{\theta, p, m_i} - \frac{l_{m_k}}{\theta}\left(\frac{\partial \theta}{\partial p}\right)_{m_j, e, n} + \left(\frac{\partial \mu_k}{\partial m_h}\right)_{\theta, p, m_j}\left(\frac{\partial m_h}{\partial p}\right)_{m_j, e, n}$$

---

where

$$\left(\frac{\partial \theta}{\partial p}\right)_{m_j, e, n} = \frac{+\,\theta\left[\mu_h - p\left(\frac{\partial v}{\partial m_h}\right)_{\theta, p, m_j}\right]\left(\frac{\partial v}{\partial \theta}\right)_{p, m_a} - p\, l_{m_h}\left(\frac{\partial v}{\partial p}\right)_{\theta, m_a}}{\left[\mu_h - p\left(\frac{\partial v}{\partial m_h}\right)_{\theta, p, m_j}\right](\Sigma\, m_a)\, c_p + p\, l_{m_h}\left(\frac{\partial v}{\partial \theta}\right)_{p, m_a}}$$

and

$$\left(\frac{\partial m_h}{\partial p}\right)_{m_j, e, n} = \frac{\theta\, p\left(\frac{\partial v}{\partial \theta}\right)^2_{p, m_a} + p\,(\Sigma\, m_a)\, c_p\left(\frac{\partial v}{\partial p}\right)_{\theta, m_a}}{\left[\mu_h - p\left(\frac{\partial v}{\partial m_h}\right)_{\theta, p, m_j}\right](\Sigma\, m_a)\, c_p + p\, l_{m_h}\left(\frac{\partial v}{\partial \theta}\right)_{p, m_a}}$$

* m_a denotes all the component masses; m_j all except m_h; m_i all except m_k.

*Group 28**

$\mathbf{m_j}, \boldsymbol{\varepsilon}, \zeta$ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\mathrm{m_j, e, z}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,\mathrm{m_a}} + \left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}} \left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j, e, z}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}} \left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j, e, z}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\mathrm{m_j, e, z}} = \left(\frac{\partial^2 \mathrm{v}}{\partial p^2}\right)_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{p,\,\mathrm{m_a}} \left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j, e, z}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}} \left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j, e, z}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\mathrm{m_j, e, z}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{p,\,\mathrm{m_a}} \left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j, e, z}} +$$
$$\left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,p,\,\mathrm{m_j}} \left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j, e, z}}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\mathrm{m_j, e, z}} = -\,\theta\left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}} + (\Sigma\,\mathrm{m_a})\left(\frac{\partial\, c_p}{\partial \theta}\right)_{p,\,\mathrm{m_a}} \left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j, e, z}} + \left[\frac{\partial}{\partial \mathrm{m_h}}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_j}} \left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j, e, z}}$$

$$\left[\frac{\partial\, l_{m\mathrm{k}}}{\partial p}\right]_{\mathrm{m_j, e, z}} = -\,\theta\left[\frac{\partial}{\partial \mathrm{m_k}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_i}} + \left\{\frac{l_{m\mathrm{k}}}{\theta} + \left[\frac{\partial}{\partial \mathrm{m_k}}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_i}}\right\}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j, e, z}} + \left(\frac{\partial\, l_{m\mathrm{k}}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}} \left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j, e, z}}$$

$$\left[\frac{\partial \mu_k}{\partial p}\right]_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{z}} = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i} - \frac{l_{m_k}}{\theta}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{z}} + \left(\frac{\partial \mu_k}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{z}}$$

where

$$\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{z}} = \frac{\mu_h\left[\theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a} + p\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right] - \mathbf{v}\,p\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathbf{m}_j} + \mathbf{v}\,(\mu_h + l_{m_h})}{\mu_h\left[(\Sigma\,\mathbf{m}_a)\,c_p - p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a} + \mathbf{n}\right] + \mathbf{n}\,l_{m_h} - p\,\mathbf{n}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathbf{m}_j}}$$

and

$$\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{z}} = \frac{\mathbf{n}\left[\theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a} + p\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right] - \mathbf{v}\left[(\Sigma\,\mathbf{m}_a)\,c_p - p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]}{\mu_h\left[(\Sigma\,\mathbf{m}_a)\,c_p - p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a} + \mathbf{n}\right] + \mathbf{n}\,l_{m_h} - p\,\mathbf{n}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathbf{m}_j}}$$

* $\mathbf{m}_a$ denotes all the component masses; $\mathbf{m}_j$ all except $\mathbf{m}_h$; $\mathbf{m}_i$ all except $\mathbf{m}_k$.

*Group 29**

$\mathbf{m}_j$, ε, χ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{x}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,\mathbf{m}_a} + \left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\,\mathbf{m}_a}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{x}} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{x}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{x}} = \left(\frac{\partial^2 \mathbf{v}}{\partial p^2}\right)_{\theta,\,\mathbf{m}_a} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{p,\,\mathbf{m}_a}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{x}} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{x}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{x}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\theta,\,\mathbf{m}_a} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{p,\,\mathbf{m}_a}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{x}} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{x}}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\, \mathbf{m}_a)\, c_p\right]_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{x}} = -\,\theta\left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\,\mathbf{m}_a} + (\Sigma\, \mathbf{m}_a)\left(\frac{\partial\, c_p}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{x}} + \left[\frac{\partial}{\partial \mathbf{m}_h}(\Sigma\, \mathbf{m}_a)\, c_p\right]_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{x}}$$

$$\left[\frac{\partial\, l_{m_k}}{\partial p}\right]_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{x}} = -\,\theta\left[\frac{\partial}{\partial \mathbf{m}_k}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_i} + \left\{\frac{l_{m_k}}{\theta} + \left[\frac{\partial}{\partial \mathbf{m}_k}(\Sigma\, \mathbf{m}_a)\, c_p\right]_{\theta,\,p,\,\mathbf{m}_i}\right\}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{x}} + \left(\frac{\partial\, l_{m_k}}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{e},\,\mathbf{x}}$$

$$\left[\frac{\partial \mu_k}{\partial p}\right]_{m_j, e, x} = \left(\frac{\partial v}{\partial m_k}\right)_{\theta, p, m} - \frac{l_{m_k}}{\theta}\left(\frac{\partial \theta}{\partial p}\right)_{m_j, e, x} + \left(\frac{\partial \mu_k}{\partial m_h}\right)_{\theta, p, m_j}\left(\frac{\partial m_h}{\partial p}\right)_{m_j, e, x}$$

---

where

$$\left(\frac{\partial \theta}{\partial p}\right)_{m_j, e, x} = \frac{(\mu_h + l_{m_h})\left[p\left(\frac{\partial v}{\partial p}\right)_{\theta, m_a} + v\right] + p\left[\theta\left(\frac{\partial v}{\partial \theta}\right)_{p, m_a} - v\right]\left(\frac{\partial v}{\partial m_h}\right)_{\theta, p, m_j}}{-(\mu_h + l_{m_h})\, p\left(\frac{\partial v}{\partial \theta}\right)_{p, m_a} + p\,(\Sigma\, m_a)\, c_p\left(\frac{\partial v}{\partial m_h}\right)_{\theta, p, m_j}}$$

and

$$\left(\frac{\partial m_h}{\partial p}\right)_{m_j, e, x} = \frac{-p\left(\frac{\partial v}{\partial \theta}\right)_{p, m_a}\left[\theta\left(\frac{\partial v}{\partial \theta}\right)_{p, m_a} - v\right] - (\Sigma\, m_a)\, c_p\left[p\left(\frac{\partial v}{\partial p}\right)_{\theta, m_a} + v\right]}{-(\mu_h + l_{m_h})\, p\left(\frac{\partial v}{\partial \theta}\right)_{p, m_a} + p\,(\Sigma\, m_a)\, c_p\left(\frac{\partial v}{\partial m_h}\right)_{\theta, p, m_j}}$$

* m_a denotes all the component masses; m_j all except m_h; m_i all except m_k.

*Group 30**

$\mathrm{m_j}$, ε, ψ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\mathrm{m_j,\,e,\,y}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,\mathrm{m_a}} + \left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,e,\,y}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,e,\,y}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\mathrm{m_j,\,e,\,y}} = \left(\frac{\partial^2 \mathrm{v}}{\partial p^2}\right)_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,e,\,y}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,e,\,y}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\mathrm{m_j,\,e,\,y}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,e,\,y}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,e,\,y}}$$

$$\left[\frac{\partial}{\partial p}\,(\Sigma\,\mathrm{m_a})\,c_p\right]_{\mathrm{m_j,\,e,\,y}} = -\,\theta\left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}} + (\Sigma\,\mathrm{m_a})\left(\frac{\partial\,c_p}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,e,\,y}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\,(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,e,\,y}}$$

$$\left[\frac{\partial\,l_{m\mathrm{k}}}{\partial p}\right]_{\mathrm{m_j,\,e,\,y}} = -\,\theta\left[\frac{\partial}{\partial \mathrm{m_k}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_i}} + \left\{\frac{l_{m\mathrm{k}}}{\theta} + \left[\frac{\partial}{\partial \mathrm{m_k}}\,(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_i}}\right\}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,e,\,y}} + \left(\frac{\partial\,l_{m\mathrm{k}}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,e,\,y}}$$

$$\left[\frac{\partial \mu_k}{\partial p}\right]_{m_j,\,e,\,y} = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,m_i} - \frac{l_{m_k}}{\theta}\left(\frac{\partial \theta}{\partial p}\right)_{m_j,\,e,\,y} + \left(\frac{\partial \mu_k}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,m_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{m_j,\,e,\,y}$$

where

$$\left(\frac{\partial \theta}{\partial p}\right)_{m_j,\,e,\,y} = \frac{\left[\mu_h - p\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,m_j}\right]\theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,m_a} - p\,l_{m_h}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,m_a}}{\left[\mathfrak{n} + (\Sigma\,\mathbf{m}_a)\,c_p\right]\left[\mu_h - p\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,m_j}\right] + l_{m_h}\left[\mathfrak{n} + p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,m_a}\right]}$$

and

$$\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{m_j,\,e,\,y} = \frac{\mathfrak{n}\left[\theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,m_a} + p\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,m_a}\right] + p\left[\theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2_{p,\,m_a} + (\Sigma\,\mathbf{m}_a)\,c_p\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,m_a}\right]}{\left[\mathfrak{n} + (\Sigma\,\mathbf{m}_a)\,c_p\right]\left[\mu_h - p\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,m_j}\right] + l_{m_h}\left[\mathfrak{n} + p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,m_a}\right]}$$

* $\mathbf{m}_a$ denotes all the component masses; $\mathbf{m}_j$ all except $\mathbf{m}_h$; $\mathbf{m}_i$ all except $\mathbf{m}_k$.

*Group 31**

$\mathbf{m}_j$, $\mathbf{n}$, ζ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\mathrm{m_j,\,n,\,z}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,\mathrm{m_a}} + \left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}} \left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,n,\,z}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}} \left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,n,\,z}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\mathrm{m_j,\,n,\,z}} = \left(\frac{\partial^2 \mathrm{v}}{\partial p^2}\right)_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{p,\,\mathrm{m_a}} \left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,n,\,z}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}} \left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,n,\,z}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\mathrm{m_j,\,n,\,z}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{p,\,\mathrm{m_a}} \left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,n,\,z}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,p,\,\mathrm{m_j}} \left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,n,\,z}}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\mathrm{m_j,\,n,\,z}} = -\theta\left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}} + (\Sigma\,\mathrm{m_a})\left(\frac{\partial\,c_p}{\partial \theta}\right)_{p,\,\mathrm{m_a}} \left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,n,\,z}} + \left[\frac{\partial}{\partial \mathrm{m_h}}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_j}} \left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,n,\,z}}$$

$$\left[\frac{\partial\,l_{m_k}}{\partial p}\right]_{\mathrm{m_j,\,n,\,z}} = -\theta\left[\frac{\partial}{\partial \mathrm{m_k}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_i}} + \left\{\frac{l_{m_k}}{\theta} + \left[\frac{\partial}{\partial \mathrm{m_k}}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_i}}\right\}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,n,\,z}} + \left(\frac{\partial\,l_{m_k}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}} \left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,n,\,z}}$$

$$\left[\frac{\partial \mu_k}{\partial p}\right]_{m_j, n, z} = \left(\frac{\partial v}{\partial m_k}\right)_{\theta, p, m_i} - \frac{l_{m_k}}{\theta}\left(\frac{\partial \theta}{\partial p}\right)_{m_j, n, z} + \left(\frac{\partial \mu_k}{\partial m_h}\right)_{\theta, p, m_j}\left(\frac{\partial m_h}{\partial p}\right)_{m_j, n, z}$$

where

$$\left(\frac{\partial \theta}{\partial p}\right)_{m_j, n, z} = \frac{\mu_h \left(\frac{\partial v}{\partial \theta}\right)_{p, m_a} + v\,\frac{l_{m_h}}{\theta}}{\frac{1}{\theta}\left[\mu_h\,(\Sigma\, m_a)\, c_p + n\, l_{m_h}\right]}$$

and

$$\left(\frac{\partial m_h}{\partial p}\right)_{m_j, n, z} = \frac{\theta\, n \left(\frac{\partial v}{\partial \theta}\right)_{p, m_a} - v\,(\Sigma\, m_a)\, c_p}{\mu_h\,(\Sigma\, m_a)\, c_p + n\, l_{m_h}}$$

* m_a denotes all the component masses; m_j all except m_h; m_i all except m_k.

*Group 32**

$\mathbf{m}_j, \mathbf{n}, \chi$ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\mathrm{m_j,\,n,\,x}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,\mathrm{m_a}} + \left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,n,\,x}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,n,\,x}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\mathrm{m_j,\,n,\,x}} = \left(\frac{\partial^2 \mathrm{v}}{\partial p^2}\right)_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,n,\,x}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,n,\,x}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\mathrm{m_j,\,n,\,x}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,n,\,x}} +$$
$$\left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,n,\,x}}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\mathrm{m_j,\,n,\,x}} = -\,\theta\left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}} + (\Sigma\,\mathrm{m_a})\left(\frac{\partial\,c_p}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,n,\,x}} + \left[\frac{\partial}{\partial \mathrm{m_h}}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,n,\,x}}$$

$$\left[\frac{\partial\,l_{m\mathrm{k}}}{\partial p}\right]_{\mathrm{m_j,\,n,\,x}} = -\,\theta\left[\frac{\partial}{\partial \mathrm{m_k}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_i}} + \left\{\frac{l_{m\mathrm{k}}}{\theta} + \left[\frac{\partial}{\partial \mathrm{m_k}}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_i}}\right\}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,n,\,x}} + \left(\frac{\partial\,l_{m\mathrm{k}}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,n,\,x}}$$

$$\left[\frac{\partial \mu_{\mathrm{k}}}{\partial p}\right]_{\mathbf{m}_{\mathrm{j}},\,\mathbf{n},\,\mathbf{x}} = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_{\mathrm{k}}}\right)_{\theta,\,p,\,\mathbf{m}_{\mathrm{i}}} - \frac{l_{m_{\mathrm{k}}}}{\theta}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_{\mathrm{j}},\,\mathbf{n},\,\mathbf{x}} + \left(\frac{\partial \mu_{\mathrm{k}}}{\partial \mathbf{m}_{\mathrm{h}}}\right)_{\theta,\,p,\,\mathbf{m}_{\mathrm{j}}}\left(\frac{\partial \mathbf{m}_{\mathrm{h}}}{\partial p}\right)_{\mathbf{m}_{\mathrm{j}},\,\mathbf{n},\,\mathbf{x}}$$

where

$$\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_{\mathrm{j}},\,\mathbf{n},\,\mathbf{x}} = \frac{\mu_{\mathrm{h}}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_{\mathrm{a}}} + \frac{\mathbf{v}}{\theta} l_{m_{\mathrm{h}}}}{\mu_{\mathrm{h}}\,(\Sigma\,\mathbf{m}_{\mathrm{a}})\,\frac{c_p}{\theta}}$$

$$\left(\frac{\partial \mathbf{m}_{\mathrm{h}}}{\partial p}\right)_{\mathbf{m}_{\mathrm{j}},\,\mathbf{n},\,\mathbf{x}} = -\frac{\mathbf{v}}{\mu_{\mathrm{h}}}$$

* m_{a} denotes all the component masses; $\mathbf{m}_{\mathrm{j}}$ all except $\mathbf{m}_{\mathrm{h}}$; $\mathbf{m}_{\mathrm{i}}$ all except $\mathbf{m}_{\mathrm{k}}$.

*Group 33**

$\mathbf{m}_j$, $\mathbf{n}$, ψ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\mathrm{m_j,\,n,\,y}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,\mathrm{m_a}} + \left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}} \left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,n,\,y}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}} \left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,n,\,y}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\mathrm{m_j,\,n,\,y}} = \left(\frac{\partial^2 \mathrm{v}}{\partial p^2}\right)_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{p,\,\mathrm{m_a}} \left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,n,\,y}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}} \left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,n,\,y}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\mathrm{m_j,\,n,\,y}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{p,\,\mathrm{m_a}} \left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,n,\,y}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,p,\,\mathrm{m_j}} \left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,n,\,y}}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\mathrm{m_j,\,n,\,y}} = -\,\theta\left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}} + (\Sigma\,\mathrm{m_a})\left(\frac{\partial\,c_p}{\partial \theta}\right)_{p,\,\mathrm{m_a}} \left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,n,\,y}} + \left[\frac{\partial}{\partial \mathrm{m_h}}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_j}} \left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,n,\,y}}$$

$$\left[\frac{\partial\,l_{m_k}}{\partial p}\right]_{\mathrm{m_j,\,n,\,y}} = -\,\theta\left[\frac{\partial}{\partial \mathrm{m_k}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_i}} + \left\{\frac{l_{m_k}}{\theta} + \left[\frac{\partial}{\partial \mathrm{m_k}}(\Sigma\,\mathrm{m_a})\,c_p\right]_{\theta,\,p,\,\mathrm{m_i}}\right\}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,n,\,y}} + \left(\frac{\partial\,l_{m_k}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}} \left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,n,\,y}}$$

$$\left[\frac{\partial \mu_k}{\partial p}\right]_{\mathbf{m}_j, \mathbf{n}, \mathbf{y}} = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta, p, \mathbf{m}_i} - \frac{l_{m_k}}{\theta}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j, \mathbf{n}, \mathbf{y}} + \left(\frac{\partial \mu_k}{\partial \mathbf{m}_h}\right)_{\theta, p, \mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j, \mathbf{n}, \mathbf{y}}$$

---

where

$$\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j, \mathbf{n}, \mathbf{y}} = \frac{\theta\left[\mu_h - p\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)_{\theta, p, \mathbf{m}_j}\right]\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p, \mathbf{m}_a} - p\, l_{m_h}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta, \mathbf{m}_a}}{\left[\mu_h - p\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)_{\theta, p, \mathbf{m}_j}\right](\Sigma\, \mathbf{m}_a)\, c_p + l_{m_h}\left[\mathbf{n} + p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p, \mathbf{m}_a}\right]}$$

and

$$\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j, \mathbf{n}, \mathbf{y}} = \frac{\theta\, p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)^2_{p, \mathbf{m}_a} + p\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta, \mathbf{m}_a}(\Sigma\, \mathbf{m}_a)\, c_p + \theta\, \mathbf{n}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p, \mathbf{m}_a}}{\left[\mu_h - p\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)_{\theta, p, \mathbf{m}_j}\right](\Sigma\, \mathbf{m}_a)\, c_p + l_{m_h}\left[\mathbf{n} + p\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p, \mathbf{m}_a}\right]}$$

* $\mathbf{m}_a$ denotes all the component masses; $\mathbf{m}_j$ all except $\mathbf{m}_h$; $\mathbf{m}_i$ all except $\mathbf{m}_k$.

*Group 34**

$\mathbf{m}_j$, ζ, χ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\mathbf{m}_j,\,\mathbf{z},\,\mathbf{x}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,\mathbf{m}_a} + \left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\,\mathbf{m}_a}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{z},\,\mathbf{x}} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{z},\,\mathbf{x}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{\mathbf{m}_j,\,\mathbf{z},\,\mathbf{x}} = \left(\frac{\partial^2 \mathbf{v}}{\partial p^2}\right)_{\theta,\,\mathbf{m}_a} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{p,\,\mathbf{m}_a}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{z},\,\mathbf{x}} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{z},\,\mathbf{x}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\mathbf{m}_j,\,\mathbf{z},\,\mathbf{x}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\theta,\,\mathbf{m}_a} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{p,\,\mathbf{m}_a}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{z},\,\mathbf{x}} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{z},\,\mathbf{x}}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\, \mathbf{m}_a)\, c_p\right]_{\mathbf{m}_j,\,\mathbf{z},\,\mathbf{x}} = -\,\theta\left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\,\mathbf{m}_a} + (\Sigma\, \mathbf{m}_a)\left(\frac{\partial c_p}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{z},\,\mathbf{x}} + \left[\frac{\partial}{\partial \mathbf{m}_h}(\Sigma\, \mathbf{m}_a)\, c_p\right]_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{z},\,\mathbf{x}}$$

$$\left[\frac{\partial\, l_{m_k}}{\partial p}\right]_{\mathbf{m}_j,\,\mathbf{z},\,\mathbf{x}} = -\,\theta\left[\frac{\partial}{\partial \mathbf{m}_k}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_i} + \left\{\frac{l_{m_k}}{\theta} + \left[\frac{\partial}{\partial \mathbf{m}_k}(\Sigma\, \mathbf{m}_a)\, c_p\right]_{\theta,\,p,\,\mathbf{m}_i}\right\}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{z},\,\mathbf{x}} + \left(\frac{\partial\, l_{m_k}}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathbf{z},\,\mathbf{x}}$$

$$\left[\frac{\partial \mu_{\mathbf{k}}}{\partial p}\right]_{\mathbf{m_j},\,\mathbf{z},\,\mathbf{x}} = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{m_k}}\right)_{\theta,\,p,\,\mathbf{m_i}} - \frac{l_{m_k}}{\theta}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m_j},\,\mathbf{z},\,\mathbf{x}} + \left(\frac{\partial \mu_{\mathbf{k}}}{\partial \mathbf{m_h}}\right)_{\theta,\,p,\,\mathbf{m_j}}\left(\frac{\partial \mathbf{m_h}}{\partial p}\right)_{\mathbf{m_j},\,\mathbf{z},\,\mathbf{x}}$$

---

where

$$\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m_j},\,\mathbf{z},\,\mathbf{x}} = \frac{\theta\,\mu_{\mathbf{h}}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m_a}} + \mathbf{v}\,l_{m_h}}{\mu_{\mathbf{h}}\,[(\Sigma\,\mathbf{m_a})\,c_p + \mathfrak{n}] + \mathfrak{n}\,l_{m_h}}$$

and

$$\left(\frac{\partial \mathbf{m_h}}{\partial p}\right)_{\mathbf{m_j},\,\mathbf{z},\,\mathbf{x}} = \frac{-\,\mathbf{v}\,(\Sigma\,\mathbf{m_a})\,c_p - \mathfrak{n}\left[\mathbf{v} - \theta\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m_a}}\right]}{\mu_{\mathbf{h}}\,[(\Sigma\,\mathbf{m_a})\,c_p + \mathfrak{n}] + \mathfrak{n}\,l_{m_h}}$$

* $\mathbf{m_a}$ denotes all the component masses; $\mathbf{m_j}$ all except $\mathbf{m_h}$; $\mathbf{m_i}$ all except $\mathbf{m_k}$.

*Group 35**

$\mathbf{m}_j$, ζ, ψ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\mathrm{m_j,\,z,\,y}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,\mathrm{m_a}} + \left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,z,\,y}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,z,\,y}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\mathrm{m_j,\,z,\,y}} = \left(\frac{\partial^2 \mathrm{v}}{\partial p^2}\right)_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,z,\,y}} + \left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial p}\right)_{\theta,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,z,\,y}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\mathrm{m_j,\,z,\,y}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,\mathrm{m_a}} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,z,\,y}} +$$
$$\left[\frac{\partial}{\partial \mathrm{m_h}}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{m_k}}\right)_{\theta,\,p,\,\mathrm{m_i}}\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,z,\,y}}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\, \mathrm{m_a})\, c_p\right]_{\mathrm{m_j,\,z,\,y}} = -\,\theta\left(\frac{\partial^2 \mathrm{v}}{\partial \theta^2}\right)_{p,\,\mathrm{m_a}} + (\Sigma\, \mathrm{m_a})\left(\frac{\partial\, c_p}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,z,\,y}} + \left[\frac{\partial}{\partial \mathrm{m_h}}(\Sigma\, \mathrm{m_a})\, c_p\right]_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,z,\,y}}$$

$$\left[\frac{\partial\, l_{m\mathrm{k}}}{\partial p}\right]_{\mathrm{m_j,\,z,\,y}} = -\,\theta\left[\frac{\partial}{\partial \mathrm{m_k}}\left(\frac{\partial \mathrm{v}}{\partial \theta}\right)_{p,\,\mathrm{m_a}}\right]_{\theta,\,p,\,\mathrm{m_i}} + \left\{\frac{l_{m\mathrm{k}}}{\theta} + \left[\frac{\partial}{\partial \mathrm{m_k}}(\Sigma\, \mathrm{m_a})\, c_p\right]_{\theta,\,p,\,\mathrm{m_i}}\right\}\left(\frac{\partial \theta}{\partial p}\right)_{\mathrm{m_j,\,z,\,y}} + \left(\frac{\partial\, l_{m\mathrm{k}}}{\partial \mathrm{m_h}}\right)_{\theta,\,p,\,\mathrm{m_j}}\left(\frac{\partial \mathrm{m_h}}{\partial p}\right)_{\mathrm{m_j,\,z,\,y}}$$

$$\left[\frac{\partial \mu_k}{\partial p}\right]_{\mathbf{m}_j, z, y} = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta, p, \mathbf{m}_i} - \frac{l_{m_k}}{\theta}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j, z, y} + \left(\frac{\partial \mu_k}{\partial \mathbf{m}_h}\right)_{\theta, p, \mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j, z, y}$$

---

where

$$\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j, z, y} = \frac{-\mu_h\left[p\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta, \mathbf{m}_a} + \mathbf{v}\right] + p\,\mathbf{v}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)_{\theta, p, \mathbf{m}_j}}{p\,\mu_h\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p, \mathbf{m}_a} + p\,\mathbf{n}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)_{\theta, p, \mathbf{m}_j}}$$

and

$$\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j, z, y} = \frac{-p\left[\mathbf{v}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p, \mathbf{m}_a} + \mathbf{n}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta, \mathbf{m}_a}\right] - \mathbf{v}\,\mathbf{n}}{p\,\mu_h\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p, \mathbf{m}_a} + p\,\mathbf{n}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_h}\right)_{\theta, p, \mathbf{m}_j}}$$

* $\mathbf{m}_a$ denotes all the component masses; $\mathbf{m}_j$ all except $\mathbf{m}_h$; $\mathbf{m}_i$ all except $\mathbf{m}_k$.

*Group 36**

$\mathbf{m}_j$, χ, ψ constant.

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\mathbf{m}_j,\,\mathrm{x},\,\mathrm{y}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,\mathbf{m}_a} + \left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\,\mathbf{m}_a}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathrm{x},\,\mathrm{y}} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathrm{x},\,\mathrm{y}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{\mathbf{m}_j,\,\mathrm{x},\,\mathrm{y}} = \left(\frac{\partial^2 \mathbf{v}}{\partial p^2}\right)_{\theta,\,\mathbf{m}_a} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{p,\,\mathbf{m}_a}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathrm{x},\,\mathrm{y}} + \left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial p}\right)_{\theta,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathrm{x},\,\mathrm{y}}$$

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\mathbf{m}_j,\,\mathrm{x},\,\mathrm{y}} = \left[\frac{\partial}{\partial p}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\theta,\,\mathbf{m}_a} + \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{p,\,\mathbf{m}_a}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_i,\,\mathrm{x},\,\mathrm{y}} +$$

$$\left[\frac{\partial}{\partial \mathbf{m}_h}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{m}_k}\right)_{\theta,\,p,\,\mathbf{m}_i}\right]_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathrm{x},\,\mathrm{y}}$$

$$\left[\frac{\partial}{\partial p}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{\mathbf{m}_j,\,\mathrm{x},\,\mathrm{y}} = -\,\theta\left(\frac{\partial^2 \mathbf{v}}{\partial \theta^2}\right)_{p,\,\mathbf{m}_a} + (\Sigma\,\mathbf{m}_a)\left(\frac{\partial c_p}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathrm{x},\,\mathrm{y}} + \left[\frac{\partial}{\partial \mathbf{m}_h}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathrm{x},\,\mathrm{y}}$$

$$\left[\frac{\partial l_{mk}}{\partial p}\right]_{\mathbf{m}_j,\,\mathrm{x},\,\mathrm{y}} = -\,\theta\left[\frac{\partial}{\partial \mathbf{m}_k}\left(\frac{\partial \mathbf{v}}{\partial \theta}\right)_{p,\,\mathbf{m}_a}\right]_{\theta,\,p,\,\mathbf{m}_i} + \left\{\frac{l_{mk}}{\theta} + \left[\frac{\partial}{\partial \mathbf{m}_k}(\Sigma\,\mathbf{m}_a)\,c_p\right]_{\theta,\,p,\,\mathbf{m}_i}\right\}\left(\frac{\partial \theta}{\partial p}\right)_{\mathbf{m}_j,\,\mathrm{x},\,\mathrm{y}} + \left(\frac{\partial\, l_{mk}}{\partial \mathbf{m}_h}\right)_{\theta,\,p,\,\mathbf{m}_j}\left(\frac{\partial \mathbf{m}_h}{\partial p}\right)_{\mathbf{m}_j,\,\mathrm{x},\,\mathrm{y}}$$

$$\left[\frac{\partial \mu_k}{\partial p}\right]_{m_j,\, x,\, y} = \left(\frac{\partial v}{\partial m_k}\right)_{\theta,\, p,\, m_i} - \frac{l_{mk}}{\theta}\left(\frac{\partial \theta}{\partial p}\right)_{m_j,\, x,\, y} + \left(\frac{\partial \mu_k}{\partial m_h}\right)_{\theta,\, p,\, m_j}\left(\frac{\partial m_h}{\partial p}\right)_{m_j,\, x,\, y}$$

---

where

$$\left(\frac{\partial \theta}{\partial p}\right)_{m_j,\, x,\, y} = \frac{\mu_h\left[p\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_a} + v - \theta\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right] + p\, l_{mh}\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_a} + p\left(\frac{\partial v}{\partial m_h}\right)_{\theta,\, p,\, m_j}\left[\theta\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a} - v\right]}{-(\mu_h + l_{mh})\left[n + p\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right] - (\Sigma\, m_a)\, c_p\left[\mu_h - p\left(\frac{\partial v}{\partial m_h}\right)_{\theta,\, p,\, m_j}\right]}$$

and

$$\left(\frac{\partial m_h}{\partial p}\right)_{m_j,\, x,\, y} = \frac{n\left[v - \theta\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right] + p\left[v\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a} - \theta\left(\frac{\partial v}{\partial \theta}\right)^2_{p,\, m_a} - (\Sigma\, m_a)\, c_p\left(\frac{\partial v}{\partial p}\right)_{\theta,\, m_a}\right]}{-(\mu_h + l_{mh})\left[n + p\left(\frac{\partial v}{\partial \theta}\right)_{p,\, m_a}\right] - (\Sigma\, m_a)\, c_p\left[\mu_h - p\left(\frac{\partial v}{\partial m_h}\right)_{\theta,\, p,\, m_j}\right]}$$

* m_a denotes all the component masses; m_j all except m_h; m_i all except m_k.

APPENDIX

DILUTE SOLUTION LAWS; IDEAL SOLUTIONS; DEFINITIONS OF FUGACITY AND ACTIVITY.

DILUTE SOLUTION LAWS

The dilute solution laws are important in the theory of solutions. They are therefore outlined rather briefly here in order to indicate their relation to the more general theory of thermodynamics.

If temperature and pressure remain constant we have, for a binary solution (section 50),

$$m_1 \, d\,\mu_1 + m_2 \, d\,\mu_2 = 0, \tag{1}$$

or, multiplying by $\mathbf{m}_1 + \mathbf{m}_2$,

$$\mathbf{m}_1 \, d\,\mu_1 + \mathbf{m}_2 \, d\,\mu_2 = 0. \tag{1'}$$

And since μ_1 and μ_2 with their partial derivatives are continuous functions of θ, p, m_2, we have

$$m_1 \left(\frac{\partial \mu_1}{\partial m_2}\right)_{\theta,\, p} + m_2 \left(\frac{\partial \mu_2}{\partial m_2}\right)_{\theta,\, p} = 0 \tag{2}$$

or, since $m_2 = \dfrac{\mathbf{m}_2}{\mathbf{m}_1 + \mathbf{m}_2}$,

$$\mathbf{m}_1 \left(\frac{\partial \mu_1}{\partial \mathbf{m}_2}\right)_{\theta,\, p,\, \mathbf{m}_1} + \mathbf{m}_2 \left(\frac{\partial \mu_2}{\partial \mathbf{m}_2}\right)_{\theta,\, p,\, \mathbf{m}_1} = 0 \tag{2'}$$

We assume as a physical hypothesis that

$$\lim_{m_2 \to 0} m_2 \left(\frac{\partial \mu_2}{\partial m_2}\right)_{\theta,\, p} = \frac{A\,\theta}{m_1} \tag{3}$$

or

$$\lim_{\mathbf{m}_2 \to 0} \mathbf{m}_2 \left(\frac{\partial \mu_2}{\partial \mathbf{m}_2}\right)_{\theta,\, p,\, \mathbf{m}_1} = A\,\theta \tag{3'}$$

where $0 < A < \infty$,[1] A being a constant for each substance or component.

[1] If $\mathbf{m}_2$ can have both positive and negative values (see section 21), then $A = O$.

Now since $m_2 = 1 - m_1$, we have

$$m_1 \left(\frac{\partial \mu_1}{\partial m_1}\right)_{\theta,\, p} = m_2 \left(\frac{\partial \mu_2}{\partial m_2}\right)_{\theta,\, p} \tag{4}$$

Therefore $\left(\frac{\partial \mu_1}{\partial m_1}\right)_{\theta, p}$ is continuous at and in the neighborhood of $m_1 = 1$ and hence we can choose a small positive quantity δ such that when m_2 is less than δ, we have

$$m_1 \left(\frac{\partial \mu_1}{\partial m_1}\right)_{\theta,\, p} = \frac{\mathrm{A}\,\theta}{m_1},\ m_2 < \delta \tag{5}$$

or

$$\mathbf{m}_1 \left(\frac{\partial \mu_1}{\partial \mathbf{m}_2}\right)_{\theta,\, p,\, \mathbf{m}_1} = -\,\mathrm{A}\,\theta \tag{5'}$$

Integrating (5) at constant θ and p, we have

$$\mu_1 = -\,\frac{\mathrm{A}\,\theta}{m_1} + \mathrm{K}; \tag{6}$$

and integrating (5′) at constant θ and p, we have

$$\mu_1 = -\,\mathrm{A}\,\theta\frac{\mathbf{m}_2}{\mathbf{m}_1} + \mathrm{K}'. \tag{6'}$$

Now when $m_2 = 0$, then $\mathrm{K} = \mu_1 + \mathrm{A}\,\theta = \zeta_1 + \mathrm{A}\,\theta$ and $\mathrm{K}' = \mu_1 = \zeta_1$. Thus $\mathrm{K}' = \mathrm{K} - \mathrm{A}\,\theta$, and (6) and (6′) become

$$\zeta_1\,(\theta, p) - \mu_1\,(\theta, p, m_1) = \mathrm{A}\,\theta\,\frac{(1 - m_1)}{m_1} = \mathrm{A}\,\theta\,\frac{m_2}{m_1} \tag{7}$$

and

$$\zeta_1\,(\theta, p) - \mu_1\,(\theta, p, m_1) = \mathrm{A}\,\theta\,\frac{\mathbf{m}_2}{\mathbf{m}_1} \tag{7'}$$

For the dilute component

$$m_2 \left(\frac{\partial \mu_2}{\partial m_2}\right)_{\theta,\, p} = \frac{\mathrm{A}\,\theta}{m_1} = \frac{\mathrm{A}\,\theta}{1 - m_2} \tag{8}$$

Integrating (8) at constant temperature and pressure, we have

$$\mu_2 = \mathrm{A}\,\theta \log \frac{\mathrm{B}\,m_2}{m_1} \tag{9}$$

or

$$\mu_2 = \mathrm{A}\,\theta \log \frac{\mathrm{B}\,\mathbf{m}_2}{\mathbf{m}_1}, \text{ [1]} \tag{9'}$$

[1] This is equivalent to Gibbs' equation 215.

where log denotes the Naperian logarithm and B is the constant of integration.

We see that as $\mathbf{m}_2$ approaches zero μ_2 becomes negatively infinite.

We empirically identify A with $\frac{R}{M}$, $A = \frac{R}{M}$, where R denotes the "Gas Constant", $R = 83.147 \times 10^6$ ergs per degree per mole, and M denotes the gram formula or molar weight of the substance.

IDEAL SOLUTIONS

An ideal solution is defined as one that satisfies the equation

$$N_2 \left(\frac{\partial \mu_2}{\partial N_2}\right)_{\theta,\, p} = \frac{R\,\theta}{M_2} \tag{10}$$

for all values of N_2, where N_2 denotes the mole fraction of component 2; thus, for a binary system,

$$N_2 = \frac{\dfrac{\mathbf{m}_2}{M_2}}{\dfrac{\mathbf{m}_1}{M_1} + \dfrac{\mathbf{m}_2}{M_2}}$$

where M_1 and M_2 denote thc gram formula or molar weights of components 1 and 2 respectively, and R denotes the "Gas Constant". μ_2 is expressed in ergs per gram.

Now

$$M_1 N_1 \frac{\partial \mu_1}{\partial N_2} + M_2 N_2 \frac{\partial \mu_2}{\partial N_2} = 0$$

where θ and p are constant, or

$$M_1 N_1 \left(\frac{\partial \mu_1}{\partial N_1}\right)_{\theta,\, p} = M_2 N_2 \left(\frac{\partial \mu_2}{\partial N_2}\right)_{\theta,\, p} \text{ since } N_1 + N_2 = 1 \tag{11}$$

Integrating (11) we have

$$\mu_2 (\theta, p, m_1) = \mu_2 (\theta, p, N_1) = \frac{R\,\theta}{M_2} \log N_2 + K''$$

or

$$\mu_1 (\theta, p, m_1) = \mu_1 (\theta, p, N_1) = \frac{R\,\theta}{M_1} \log N_1 + K'''$$

When $N_1 \rightarrow 1$ then $K''' = \mu_1(\theta, p, 1) = \zeta_1(\theta, p)$ and hence

$$\mu_1(\theta, p, N_1) - \zeta_1(\theta, p) = \frac{R\,\theta}{M_1} \log N_1$$

or

$$N_1 = e^{\frac{M_1 \mu_1(\theta, p, N_1) - M_1 \zeta_1(\theta, p)}{R\,\theta}}$$

In the limit as $N_1 \rightarrow 1$ then $N_2 \rightarrow 0$, and $\mu_2(\theta, p, N_1)$ becomes negatively infinite.

Now transforming from gram masses to mole fractions we have

$$\mathbf{m}_2 \left(\frac{\partial \mu_2}{\partial \mathbf{m}_2}\right)_{\theta,\, p,\, \mathbf{m}_1} = N_1 N_2 \left(\frac{\partial \mu_2}{\partial N_2}\right)_{\theta,\, p}$$

Thus in the limit as $N_2 \rightarrow 0$, $N_1 \rightarrow 1$ and hence

$$\lim_{\mathbf{m}_2 \text{ or } N_2 \rightarrow 0} \mathbf{m}_2 \frac{\partial \mu_2}{\partial \mathbf{m}_2} = N_2 \frac{\partial \mu_2}{\partial N_2} = \frac{R\,\theta}{M_2}$$

We therefore find that we can identify the dilute solution laws with the definition of an ideal solution. We see that as N_1 departs appreciably from unity we have the inequality

$$\mathbf{m}_2 \frac{\partial \mu_2}{\partial \mathbf{m}_2} \neq N_2 \frac{\partial \mu_2}{\partial N_2}$$

but the dilute solution laws are only applicable for systems in which N_2 is very small, that is, N_1 does not depart appreciably from unity.

Hence if we wished we could set down the equation

$$N_2 \left(\frac{\partial \mu_2}{\partial N_2}\right)_{\theta,\, p} = \frac{R\,\theta}{M_2},\ 0 \leqq N_2 \leqq \delta$$

as a general equation where δ is some positive value equal to or less than one, this value depending on the particular system chosen. If the solution is an ideal solution then $\delta = 1$, if not then δ is less than one and in general (dilute solutions) we know that it is usually very small.

FUGACITY AND ACTIVITY

The fugacity is defined as a function of temperature and pressure in a simple system by the equation

$$f(\theta, p) = f(\theta, p_0)\, e^{\frac{M\zeta(\theta, p) - M\zeta(\theta, p_0)}{R\theta}}*$$

or

$$M\zeta(\theta, p) - M\zeta(\theta, p_0) = R\theta \log \frac{f(\theta, p)}{f(\theta, p_0)}$$

where e denotes the base of the Naperian logarithms, R = 83.147×10^6 ergs per degree per mole, M denotes the gram formula or molar weight of the substance, $\zeta(\theta, p)$ and $\zeta(\theta, p_0)$ the zeta per unit mass at the states (θ, p) and (θ, p_0) respectively, and $f(\theta, p)$ and $f(\theta, p_0)$ the fugacities at the states (θ, p) and (θ, p_0) respectively.

To complete the definition of fugacity we must assign a value to $f(\theta, p_0)$.

Now assuming as a physical hypothesis that we are dealing only with systems such that, if we first choose a small positive quantity K to represent the maximum allowable error of our result we can pick a value $\delta > 0$ so that the characteristic equation (equation of state) of the system for values of p less than δ and greater than zero will differ from $p = \frac{vM}{R\theta}$ by less than the value K, *i.e.* $\left|\frac{vM}{R\theta} - p\right| <$ K where $0 < p < \delta$.

Then where $0 < p < \delta$ we have

$$M\left(\frac{\partial\zeta}{\partial p}\right)_\theta = Mv = \frac{R\theta}{p}$$

Integrating this expression at constant temperature we have

$$\frac{p}{p_0} = e^{\frac{M\zeta(\theta, p) - M\zeta(\theta, p_0)}{R\theta}}$$

or

$$p = K\, e^{\frac{M\zeta(\theta, p)}{R\theta}} \text{ where } K = \text{a constant.}$$

*In identifying this equation with G. N. Lewis' equation note that Lewis' F = M ζ, *i.e.* F is the Zeta (Free Energy) per mole of substance.

Thus when p approaches zero as a limit, θ being constant, $M\,\zeta(\theta, p)$ becomes negatively infinite.

Hence

$$\lim_{\substack{p \to 0 \\ \theta = K'}} f(\theta, p) = 0$$

Lewis completes the definition of fugacity by defining $f(\theta, p_0)$ as p_0 where $0 < p < \delta$, $f(\theta, p_0) = p_0$, $0 < p < \delta$.

If $f(\theta, p_0)$ were chosen as zero when $p = 0$ then ζ would become negatively infinite, giving us an indeterminate form for $f(\theta, p)$ which therefore could not be evaluated.

Now when the system is such that $M\,p\,v = R\,\theta$ then $f(\theta, p) = p$, and hence a tabulation of fugacities for gases will give the deviation of these gases from the perfect gas law.

For a liquid the fugacity of the liquid would be the fugacity of the vapor in equilibrium with it and thus, if this vapor obeyed the perfect gas law, the fugacity, $f(\theta, p)$, of the liquid would be equal to its vapor pressure at the temperature θ.

The fugacity of, let us say, component one in a binary solution of unit mass is defined as

$$f_1(\theta, p, m_1) = f_1(\theta, p_0, m_{1_0})\, e^{\frac{M_1 \mu_1(\theta, p, m_1) - M_1 \mu_1(\theta, p_0, m_{1_0})}{R\theta}}$$

To complete the definition we must assign a value to $f_1(\theta, p_0, m_{1_0})$.

The activity, a, is defined as a function of θ and p for a simple substance by the equation

$$a(\theta, p) = e^{\frac{M\,\zeta(\theta, p) - M\,\zeta(\theta, p_1)}{R\theta}}$$

or

$$\zeta(\theta, p) - \zeta(\theta, p_1) = \frac{R}{M}\theta \log a(\theta, p).$$

To complete the definition of activity we must assign a value to p_1 at each temperature.

The activity of, say, component one of a homogeneous binary solution of unit mass is defined as a function of θ, p, and m_1 by the equation

$$a_1(\theta, p, m_1) = e^{\frac{M_1 \mu_1(\theta, p, m_1) - M_1 \mu_1(\theta, p_1, m_{1_0})}{R\theta}}$$

To complete the definition we must assign a value to p_1 and m_{1_0} at each temperature.

The standard states have been defined as follows.

(1). For a gas the activity is made equal to the fugacity, $a(\theta, p) = f(\theta, p)$. In other words p_1 is so chosen that $f(\theta, p_1) = 1$ or

$$e^{\frac{M\zeta(\theta, p_1)}{R\theta}} = \frac{e^{\frac{M\zeta(\theta, p_0)}{R\theta}}}{f(\theta, p_0)}$$

(2). For a liquid or solid which may act as a solvent the activity is made equal to unity when the concentration of the substance is unity. In other words $p_1 = p$ when $m_{1_0} = 1$, or

$$a(\theta, p) = e^{\frac{M\zeta(\theta, p) - M\zeta(\theta, p_1)}{R\theta}} = e^0 = 1$$

Thus

$$a(\theta, p, m_1) = e^{\frac{M_1 \mu_1(\theta, p, m_1) - M_1 \zeta_1(\theta, p)}{R\theta}}$$

Now from the dilute solution laws or ideal solution definition we have

$$N_1 = e^{\frac{M_1 \mu_1(\theta, p, m_1) - M_1 \zeta_1(\theta, p)}{R\theta}}$$

Hence for the series of states, *i.e.* in the regions, in which a homogeneous binary system satisfies the ideal solution definition,

$$a_1(\theta, p, m_1) = N_1 \text{ where } N_1 > N_2$$

(3). For a solute. We perceive that in the region or series of states in which the ideal solution definition is satisfied the activity of the solute will be proportional to the mole fraction of the solute.

Furthermore, the limiting value of the activity, as m_2 approaches zero, will be zero,

$$\lim_{m_2 \to 0} a_2(\theta, p, m_1) = 0.$$

The standard state is so chosen that in the region in which the ideal solution definition is satisfied

$$a_2(\theta, p, m_1) = N_2$$

Thus if the solution acted as an ideal solution over the whole range of concentrations we would then have

$$a_2(\theta, p, m_1) = N_2 = e^{\frac{M_2 \mu_2(\theta, p, m_1) - M_2 \zeta_2(\theta, p)}{R\theta}}$$

INDEX